MATHEMATICS

MATHEMATICS

MATHEMATICS

MATHEMATICS

MATHEMATICS 31

MODULE 4

Trigonometry

MATHEMATICS

MATHEMATICS

MATHEMATICS

MATHEMATICS

Distance Learning

Alberta EDUCATION

Mathematics 31

Module 4

TRIGONOMETRY

Distance Learning

Alberta EDUCATION

This document is intended for	
Students	✓
Teachers (Mathematics 31)	✓
Administrators	
Parents	
General Public	
Other	

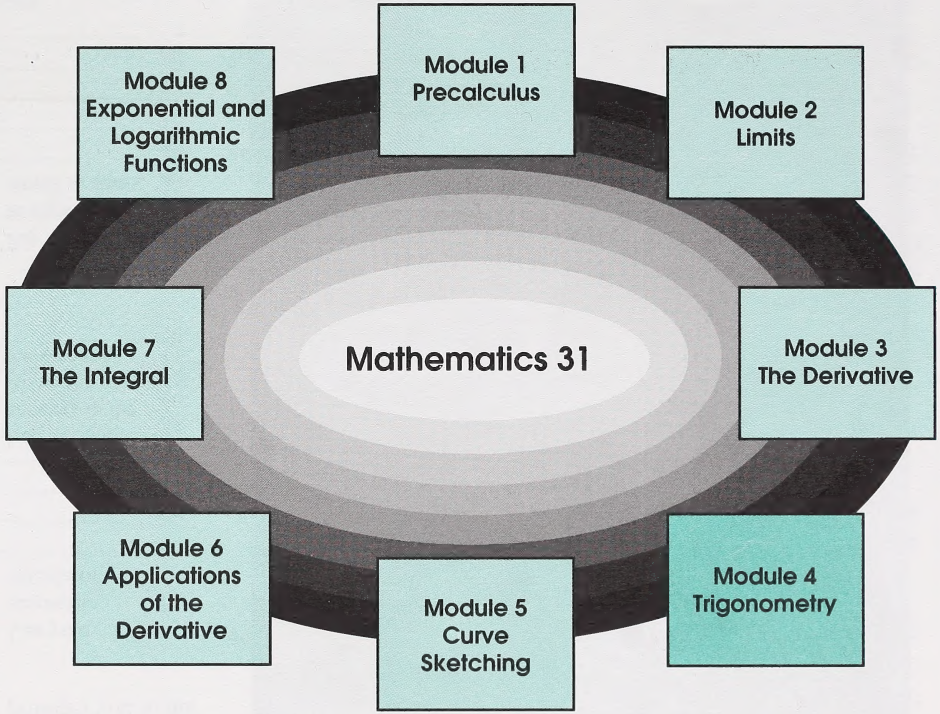
Mathematics 31
Student Module Booklet
Module 4
Trigonometry
Alberta Distance Learning Centre
ISBN 0-7741-1184-4

Welcome

WESTFILE INC.

Welcome to Module 4. We hope you'll enjoy your study of Trigonometry.

Mathematics 31 contains eight modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



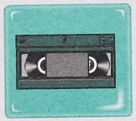
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

- Use your scientific calculator.

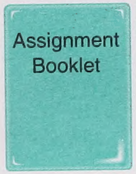
- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.



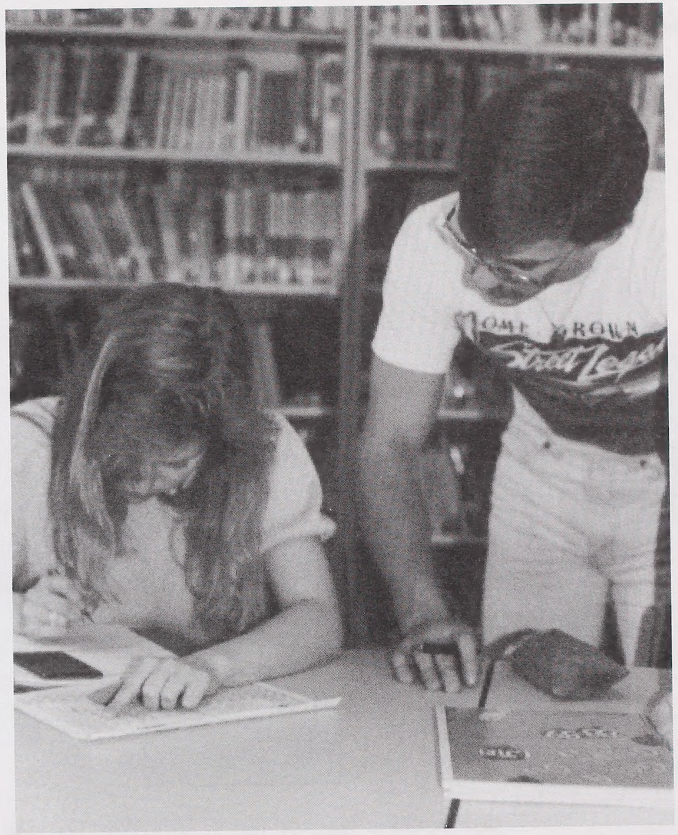
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

Module Overview 1
Evaluation 2

Section 1: Trigonometric Identities 3
Activity 1: Reciprocal, Quotient, and Pythagorean Identities 4
Activity 2: Sum and Difference Formulas 9
Activity 3: Double- and Half-Angle Identities 15
Follow-up Activities 19
Extra Help 19
Enrichment 23
Conclusion 23
Assignment 23

Section 2: Trigonometric Equations 24
Activity 1: Specific Solutions 25
Activity 2: General Solutions 33
Follow-up Activities 34
Extra Help 34
Enrichment 40
Conclusion 43
Assignment 43

Section 3: Limits of Trigonometric Functions 44
Activity 1: Sine and Cosine 45
Activity 2: Complex Trigonometric Expressions 51
Follow-up Activities 57
Extra Help 57
Enrichment 59
Conclusion 60
Assignment 60

Section 4: Derivatives of Trigonometric Functions 61
Activity 1: Primary Ratios 62
Activity 2: Reciprocal Ratios 68
Activity 3: Complex Trigonometric Expressions 70
Activity 4: Trigonometric Curves and Their Tangents 78
Follow-up Activities 83
Extra Help 83
Enrichment 86
Conclusion 87
Assignment 87

Module Summary 88
Final Module Assignment 88

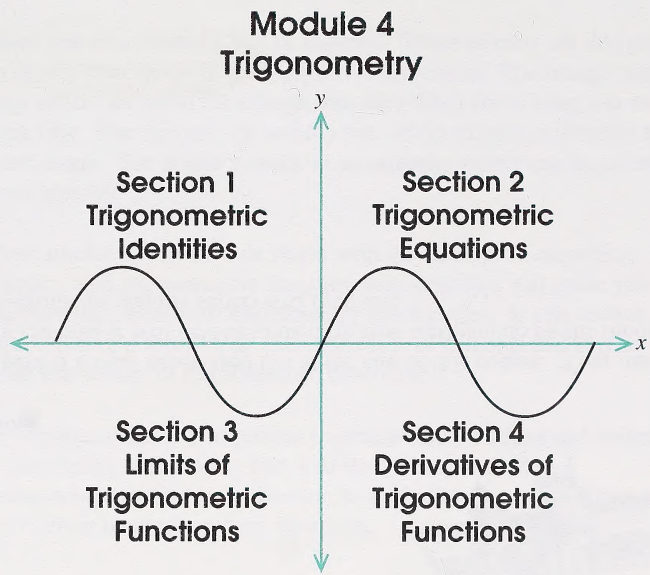
Appendix 89
Glossary 90
Suggested Answers 90

Module Overview

The height of your foot above the ground when walking, the average temperature for each day of the year, the displacement of particles in a sound wave, and the height reached by tides all repeat in a cyclical manner. Economists, traders, and stock brokers make predictions of the future based on business trends that are periodic. Builders in the last century used the principles of trigonometry to construct buildings, bridges, and tunnels. The science of engineering came about due to modern equipment and trigonometry.

In earlier courses you encountered many functions that repeat periodically. The trigonometric functions $y = \sin\theta$ and $y = \cos\theta$ are examples of two of these. The study of trigonometry and its applications have a broad history, most of which was related to astronomy, land measurement, and construction. Today, people working in the engineering, electronics, and space exploration fields rely heavily on the skills and concepts of this branch of mathematics.

In this module you will study trigonometric identities and equations, limits, and derivatives of trigonometric functions.



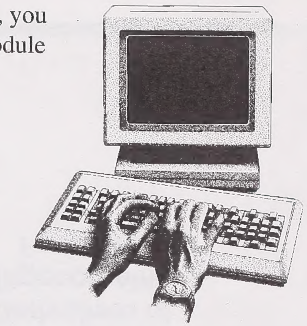
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete four section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**15 marks**
Section 2 Assignment	**13 marks**
Section 3 Assignment	**16 marks**
Section 4 Assignment	**20 marks**
Final Module Assignment	**36 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Trigonometric Identities

Have you ever visited a hall of mirrors? These mirrors are designed to distort your shape in many different directions. The images that they reflect are good for a laugh, but they don't show what you truly look like. The mirrors you usually use reflect images equivalent to your shape. The image you see in an ordinary mirror can be called your identity.

Your studies in this module begin with the concept of identities. Facility with trigonometric functions and identities will make your study of the calculus of trigonometry much easier. In this section you will take a short break from the study of calculus and extend your knowledge of trigonometric functions.

In previous courses you proved trigonometric identities and solved trigonometric equations. You will study these and other relationships further to enable you to apply the calculus of limits and derivatives to trigonometric functions.

Activity 1: Reciprocal, Quotient, and Pythagorean Identities

In calculus, it is often helpful to change trigonometric expressions from one form to an equivalent form that is easier to work with.

An equation that equates the two equivalent trigonometric expressions is called a **trigonometric identity**.

Note: The measure of angles in this module will be confined to radian measure, since it facilitates the simplification of the derivatives of trigonometric functions.

You have encountered most of the fundamental trigonometric identities in previous courses.

Reciprocal and Quotient Identities

The reciprocal identities are as follows:

- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$

The quotient identities are as follows:

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Reciprocal and quotient identities allow you to rewrite any of the other four ratios in terms of sine and cosine.

Example 1

Simplify $\left(\cos^2\theta\right)(\sec\theta)(\tan\theta)$.

Solution

Substitute equivalent expressions for $\sec\theta$ and $\tan\theta$. Remember that $\cos^2\theta$ means $(\cos\theta)(\cos\theta)$.

$$\begin{aligned}\cos^2\theta(\sec\theta)(\tan\theta) &= \left(\cos^2\theta\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \sin\theta\end{aligned}$$

Example 2

Simplify $\dfrac{\cot\theta\ \cos\theta}{\csc\theta}$.

Solution

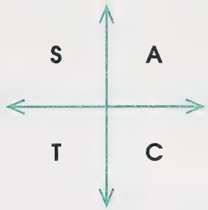
$$\begin{aligned}\frac{\cot\theta\ \cos\theta}{\csc\theta} &= \frac{\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{\cos\theta}{1}\right)}{\frac{1}{\sin\theta}}\\ &= \cos^2\theta\end{aligned}$$

1. Simplify the following trigonometric expressions.

a. $(\sin^2 \theta)(\csc \theta)(\cot \theta)$ **b.** $\dfrac{\tan \theta \, \sin \theta}{\sec \theta}$

Check your answers by turning to the Appendix.

Recall from previous courses the **CAST rule**. This rule is a quick method of determining the signs of trigonometric functions in the various quadrants.



The CAST rule shows you the pair of quadrants in which a primary trigonometric ratio and its reciprocal are positive or negative. For example, cosine and secant are positive in the first and fourth quadrants, and sine and cosecant are negative in the third and fourth quadrants.

Also, recall that every trigonometric function of θ has the same value as the trigonometric cofunction of $(90° - \theta)$. You can remember this if you always think of $(90° - \theta)$ as the other angle.

It is interesting to note that the words *cosine*, *cosecant*, and *cotangent* are short forms for the sine of the complementary angle, the secant of the complementary angle, and the tangent of the complementary angle.

∴ cosine	means	the sine of the other angle
cosecant	means	the secant of the other angle
cotangent	means	the tangent of the other angle

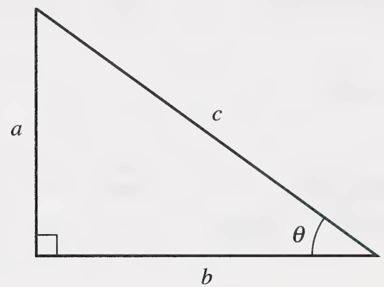
This relationship gives six **cofunction identities.**

- $\sin \theta = \cos (90° - \theta)$
- $\tan \theta = \cot (90° - \theta)$
- $\sec \theta = \csc (90° - \theta)$
- $\cos \theta = \sin (90° - \theta)$
- $\csc \theta = \sec (90° - \theta)$
- $\cot \theta = \tan (90° - \theta)$

Pythagorean Identities

The basic Pythagorean identity is $\sin^2 \theta + \cos^2 \theta = 1$.

Using a right triangle, it can be proven.



Remember that $\sin^2\theta = (\sin\theta)^2$; therefore, $\sin^2\theta + \cos^2\theta = 1$. This can also be rewritten as follows:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$
$$\frac{a^2 + b^2}{c^2} = 1$$

Since the numerator $a^2 + b^2$ is equal to c^2, $\frac{c^2}{c^2} = 1$.

The other two forms of the Pythagorean identity can be developed from this basic form by, in turn, dividing each term by $\sin^2\theta$, and then by $\cos^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\tan^2\theta + 1 = \sec^2\theta$$

The three identities discussed are called Pythagorean identities and are used often in calculus problems that involve trigonometry. To summarize, the identities are as follows:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \csc^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$

Problems that involve changing trigonometric expressions from one form to another are written as equations. Your task is to transform one side of the equation into the other.

Although an equal sign is present, the rules for equations (adding or multiplying both sides by the same amount) are **not** used. The expressions are transformed individually.

To accomplish the transformation of one or the other or both expressions, follow these steps:

Step 1: Replace $\tan x$, $\csc x$, $\sec x$, and $\cot x$ with equivalent expressions that contain $\sin x$ and $\cos x$.

Step 2: Look for forms of the Pythagorean identity that can be replaced.

Step 3: Manipulate the expressions algebraically. Factoring, creating common denominators, and multiplying by an expression equivalent to 1 are the most common.

Step 4: Be inventive. There is more than one correct way to show that two expressions are equivalent. If one thing doesn't work, try another. With practice, these transformations will become familiar and you will accomplish them with ease.

Example 1

Show that $\sin x \tan x = \sec x - \cos x$.

Solution

LS	RS
$\sin x \tan x$	$\sec x - \cos x$
$= \sin x\left(\dfrac{\sin x}{\cos x}\right)$ (quotient identity)	$= \dfrac{1}{\cos x} - \cos x$ (reciprocal identity)
$= \dfrac{\sin^2 x}{\cos x}$	$= \dfrac{1-\cos^2 x}{\cos x}$ (common denominator)
	$= \dfrac{\sin^2 x}{\cos x}$ (Pythagorean identity)

LS = RS

Example 2

Prove that $\dfrac{\cos^2\theta}{1+\sin\theta} = 1-\sin\theta$.

Solution

LS	RS
$\dfrac{\cos^2\theta}{1+\sin\theta}$	$1-\sin\theta$
$= \dfrac{1-\sin^2\theta}{1+\sin\theta}$ (Pythagorean identity)	
$= \dfrac{(1+\sin\theta)(1-\sin\theta)}{1+\sin\theta}$ (Factor the numerator.)	
$= 1-\sin\theta$	

LS = RS

Some identity proofs require that you begin with an algebraic manipulation. The following example starts with multiplying the left side by a trigonometric expression equal to 1, thus allowing you to use one of the known identities.

Example 3

Prove the identity $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$.

Solution

LS	RS
$\dfrac{\sin x}{1+\cos x}$	$\dfrac{1-\cos x}{\sin x}$
$= \dfrac{\sin x}{1+\cos x} \bullet \dfrac{1-\cos x}{1-\cos x}$	
$= \dfrac{\sin x(1-\cos x)}{1-\cos^2 x}$	
$= \dfrac{\sin x(1-\cos x)}{\sin^2 x}$	
$= \dfrac{1-\cos x}{\sin x}$	

LS = RS

Note: The pattern of the terms $\sin x$ and $\cos x$ in Example 3 suggests a similar identity: $\frac{\cos x}{1+\sin x} = \frac{1-\sin x}{\cos x}$. This identity can be proved if you follow the same steps as in Example 3.

Example 4

Prove that $\frac{1+\tan\theta}{1+\cot\theta}=\frac{1-\tan\theta}{\cot\theta-1}$.

Solution

LS	RS	
$\frac{1+\tan\theta}{1+\cot\theta}$	$\frac{1-\tan\theta}{\cot\theta-1}$	
$=\frac{1+\frac{\sin\theta}{\cos\theta}}{1+\frac{\cos\theta}{\sin\theta}}$	$=\frac{1-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}-1}$	(Rewrite in terms of $\sin\theta$ or $\cos\theta$.)
$=\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta}}$	$=\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\sin\theta}}$	(Convert to common denominators in the numerator and the denominator.)
$=\frac{\cos\theta+\sin\theta}{\cos\theta}\bullet\frac{\sin\theta}{\sin\theta+\cos\theta}$	$=\frac{\cos\theta-\sin\theta}{\cos\theta}\bullet\frac{\sin\theta}{\cos\theta-\sin\theta}$	
$=\frac{\sin\theta}{\cos\theta}$	$=\frac{\sin\theta}{\cos\theta}$	

LS = RS

2. Prove the following identities.

a. $1-\sin^2\theta=\sin^2\theta\ \cot^2\theta$

b. $\sec x(1+\cos x)=1+\sec x$

c. $\frac{\sin\theta}{1+\cos\theta}+\frac{\sin\theta}{1-\cos\theta}=2\csc\theta$

d. $\frac{1+\sec\theta}{\tan\theta}-\csc\theta=\cot\theta$

e. $\sec^2 x\ \sec^2 y-\tan^2 x\ \sec^2 y+\tan^2 x\ \tan^2 y-\sec^2 x\ \tan^2 y=1$

3. $\dfrac{\csc x + \sec x}{\sin x + \cos x} = \csc x \sec x$

a. Prove the identity.

b. Predict a similar identity for the expression $\frac{\cos x + \cot x}{\sec x + \tan x}$, and prove that it is correct.

Check your answers by turning to the Appendix.

Activity 2: Sum and Difference Formulas

In mathematics, evaluating functions is often combined with the operations of addition and subtraction. The results are sometimes different if the operations are carried out in different orders.

Case 1: double a sum = the sum of the numbers doubled

$$2(x+y) = 2x + 2y$$

Case 2: the square of a sum ≠ the sum of squares

$$(x+y)^2 \neq x^2 + y^2$$

Case 3: the reciprocal of a sum ≠ the sum of reciprocals

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$$

Case 1 shows the **distributive property** of multiplication over addition. Cases 2 and 3 are function operations that are not distributive over addition. Sine, cosine, and tangent are functions that are not distributive over addition (or subtraction) as well.

$$\sin(x \pm y) \neq \sin x \pm \sin y$$
$$\cos(x \pm y) \neq \cos x \pm \cos y$$
$$\tan(x \pm y) \neq \tan x \pm \tan y$$

Example 1

Prove that $\sin(\alpha + \beta) \neq \sin \alpha + \sin \beta$. Let $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

Solution

LS	RS
$\sin(\alpha+\beta)$	$\sin \alpha + \sin \beta$
$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$	$= \sin \frac{\pi}{3} + \sin \frac{\pi}{6}$
$= \sin \frac{\pi}{2}$	$= \frac{\sqrt{3}}{2} + \frac{1}{2}$
$= 1$	$= \frac{\sqrt{3}+1}{2}$

LS ≠ RS

Therefore, $\sin(\alpha + \beta) \neq \sin \alpha + \sin \beta$.

This example shows you that $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$. The same is true for the other trigonometric function operations. The problem now to consider is to find identity expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$, and $\tan(\alpha \pm \beta)$.

Recall from Mathematics 30 that the **sum and difference formulas** for sine and cosine were developed. They are as follows:

- $\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$
- $\sin(\alpha-\beta) = \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta$
- $\cos(\alpha+\beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$
- $\cos(\alpha-\beta) = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$

The sum and difference identities for tangent are developed using a combination of the quotient identity for tangent and the sum identities for sine and cosine.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta}{\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta}\end{aligned}$$

There are many forms of tangent identities. The most often used are the ones in which the first entry in the denominator is the number 1. To change $\cos\alpha\ \cos\beta$ to 1, you must divide it by itself. To do this you must also divide every other term in the expression by $\cos\alpha\ \cos\beta$ so that the value of the expression remains unchanged.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\dfrac{\sin\alpha\ \cancel{\cos\beta}}{\cos\alpha\ \cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\ \sin\beta}{\cancel{\cos\alpha}\ \cos\beta}}{\dfrac{\cancel{\cos\alpha\ \cos\beta}}{\cancel{\cos\alpha\ \cos\beta}} - \dfrac{\sin\alpha\ \sin\beta}{\cos\alpha\ \cos\beta}}\\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \left(\dfrac{\sin\alpha}{\cos\alpha}\right)\left(\dfrac{\sin\beta}{\cos\beta}\right)}\end{aligned}$$

$$\therefore\ \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}$$

A similar method is used to determine the identity $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\ \tan\beta}$.

The proof is left for you to do as one of the questions.

The following examples show some of the ways these sum and difference identities are used.

Example 2

Express each of the following as a single trigonometric function.

- $\sin 8 \cos 3 + \cos 8 \sin 3$
- $\cos 5a \cos 2a - \sin 5a \sin 2a$
- $\dfrac{\tan m + \tan m}{1 - \tan^2 m}$

Solution

Using careful observation, notice that $\sin 8 \cos 3 + \cos 8 \sin 3$ models the expansion of the sum identity for sine.

$$\begin{aligned}\sin 8 \cos 3 + \cos 8 \sin 3 &= \sin (8+3)\\ &= \sin 11\end{aligned}$$

The term **sin 11** is considered to be a single trigonometric function.

The expansion of the difference of a cosine is shown in $\cos 5a \cos 2a - \sin 5a \sin 2a$.

$$\begin{aligned}\cos 5a \cos 2a - \sin 5a \sin 2a &= \cos (5a + 2a)\\ &= \cos 7a\end{aligned}$$

The tangent sum identity is modelled in $\frac{\tan m + \tan m}{1-\tan^2 m}$.

$$\frac{\tan m + \tan m}{1 - \tan^2 m} = \tan 2m$$

Example 3

Prove that $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha$.

Solution

LS	RS
$\sin\left(\dfrac{\pi}{2} + \alpha\right)$	$\cos \alpha$
$= \sin \dfrac{\pi}{2} \cos \alpha + \cos \dfrac{\pi}{2} \sin \alpha$	
$= 1 \bullet \cos \alpha + 0 \bullet \sin \alpha$	
$= \cos \alpha$	

LS = RS

Use the unit circle or a calculator to find the values of expressions like $\sin \frac{\pi}{2}$ and $\cos \frac{\pi}{2}$.

Example 4

Simplify $\dfrac{\cos(30° - x) + \sin(x - 60°)}{\cos x}$.

Solution

$$\begin{aligned}\frac{\cos(30° - x) + \sin(x - 60°)}{\cos x} &= \frac{(\cos 30° \cos x + \sin 30° \sin x) + (\sin x \cos 60° - \cos x \sin 60°)}{\cos x} \\ &= \frac{\left(\frac{\sqrt{3}}{2} \bullet \cos x + \frac{1}{2} \bullet \sin x\right) + \left(\sin x \bullet \frac{1}{2} - \cos x \bullet \frac{\sqrt{3}}{2}\right)}{\cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

1. Simplify the following expressions.

a. $\sin 2n \cos 3n - \cos 2n \sin 3n$

b. $\cos a \cos 3a - \sin a \sin 3a$

c. $\dfrac{\tan 6 + \tan 4}{1 - \tan 6 \tan 4}$

d. $\dfrac{\sin(60° + y) - \cos(y - 30°)}{\sin y}$

2. Use the quotient identity for tangent to prove the difference identity for tangent:

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

Check your answers by turning to the Appendix.

The six sum and difference identities can be used to prove other identities.

Example 5

Show that $\frac{\cos(x+y)}{\sin x \sin y} = \cot x \cot y - 1$.

Solution

Work with the left side of the equation.

$$\frac{\cos(x+y)}{\sin x \sin y} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y} \quad \text{(sum identity for cosine)}$$

$$= \frac{\cos x \cos y}{\sin x \sin y} - \frac{\sin x \sin y}{\sin x \sin y}$$

$$= \cot x \cot y - 1 \quad \text{(quotient identities for cotangent)}$$

Therefore, $\text{LS} = \text{RS}$.

Example 6

Prove the following:

$$\sec(-x)\sin\left(\frac{\pi}{2} - x\right) + \sin(-x)\cos\left(\frac{\pi}{2} - x\right) = \cos^2 x$$

Solution

Work with the left side of the equation.

$$\sec(-x)\sin\left(\frac{\pi}{2} - x\right) + \sin(-x)\cos\left(\frac{\pi}{2} - x\right)$$

$$= \sec x\left(\sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x\right)$$

$$- \sin x\left(\cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x\right)$$

$$= \sec x(1 \bullet \cos x - 0 \bullet \sin x) - \sin x(0 \bullet \cos x + 1 \bullet \sin x)$$

$$= \sec x \bullet \cos x - \sin^2 x$$

$$= 1 - \sin^2 x \quad \text{(reciprocal identities)}$$

$$= \cos^2 x \quad \text{(Pythagorean identity)}$$

Therefore, $\text{LS} = \text{RS}$.

Example 7

Prove that $\dfrac{\tan(x+y)-\tan x}{1+\tan(x+y)\tan x}=\tan y$.

Solution

Work with the left side of the equation.

$$\frac{\tan(x+y)-\tan x}{1+\tan(x+y)\tan x}=\frac{\frac{\tan x+\tan y}{1-\tan x\tan y}-\tan x}{1+\frac{\tan x+\tan y}{1-\tan x\tan y}\bullet\tan x}\qquad\text{(sum identity for tangent)}$$

$$=\frac{\frac{(\tan x+\tan y)-\tan x(1-\tan x\tan y)}{1-\tan x\tan y}}{\frac{(1-\tan x\tan y)+\tan x(\tan x+\tan y)}{1-\tan x\tan y}}$$

$$=\frac{\tan x+\tan y-\tan x+\tan^2 x\tan y}{\cancel{1-\tan x\tan y}}\bullet\frac{\cancel{1-\tan x\tan y}}{1-\tan x\tan y+\tan^2 x+\tan x\tan y}$$

$$=\frac{\tan y+\tan^2 x\tan y}{1+\tan^2 x}$$

$$=\frac{\tan y\left(\cancel{1+\tan^2 x}\right)}{\cancel{1+\tan^2 x}}$$

$$=\tan y$$

Therefore, $LS=RS$.

3. Prove the following using the sum and difference formulas.

   a. $\cos y - \tan x \sin y = \dfrac{\cos(x+y)}{\cos x}$

   b. $\cos x\left[\sin\left(\dfrac{\pi}{2}-x\right)\right] + \sin x\left[\cos\left(\dfrac{\pi}{2}-x\right)\right] = 1$

   c. $\cos(x+y)\cos(x-y) = \cos^2 y - \sin^2 x$

   d. $\tan(x+y)\tan(x-y) = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y}$

4. By writing $\sec(a+b)$ as $\frac{1}{\cos(a+b)}$, derive a formula for $\sec(a+b)$ in terms of $\sec a$, $\sec b$, $\csc a$, and $\csc b$.

Check your answers by turning to the Appendix.

To facilitate your future work in mathematics, memorize the sum and difference formulas for sine, cosine, and tangent.

Activity 3: Double- and Half-Angle Identities

In the previous activity you worked with formulas for the sine, cosine, and tangent of the sum of two angles.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

If the two angles, a and b, are equal, then these formulas become identities for $\sin 2a$, $\cos 2a$, and $\tan 2a$, called **double-angle identities**.

Double-Angle Identities

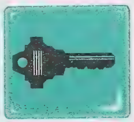
To develop the identity for $\sin 2a$, use the identity for $\sin(a+b)$ and replace b with a.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a+a) = \sin a \cos a + \cos a \sin a$$
$$\sin 2a = 2 \sin a \cos a$$

Therefore, the double-angle identity for the sine function is $\sin 2a = 2\sin a \cos a$.

The identity for $\cos 2a$ is developed by replacing the b with a in the identity for $\cos(a+b)$.

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a+a) = \cos a \cos a - \sin a \sin a$$
$$\cos 2a = \cos^2 a - \sin^2 a$$



Therefore, the double-angle identity for the cosine function is $\cos 2a = \cos^2 a - \sin^2 a$.

You can get two other identities for $\cos 2a$ by using one of the forms of the basic Pythagorean identity $\sin^2 a + \cos^2 a = 1$ and substituting. Remember that this basic identity can be written as $\sin^2 a = 1 - \cos^2 a$ or $\cos^2 a = 1 - \sin^2 a$. Therefore, $\cos 2a = \cos^2 a - \sin^2 a$.

$$\cos 2a = \left(1 - \sin^2 a\right) - \sin^2 a \qquad \cos 2a = \cos^2 a - \left(1 - \cos^2 a\right)$$
$$= 1 - 2\sin^2 a \qquad\qquad = 2\cos^2 a - 1$$

As an exercise, develop the double-angle identity for $\tan 2a$.

In calculus, you will often encounter expressions that contain $\sin x$ and $\sin 2x$ or $\cos x$ and $\cos 2x$. The double-angle identities allow you to simplify by replacing the $\sin 2x$ and $\cos 2x$ with equivalent expressions in terms of just $\sin x$ and $\cos x$.

Example 1

Show that $\dfrac{\sin x}{\sin 2x} = \dfrac{\sec x}{2}$.

Solution

Whenever you see a trigonometric function of $2x$, you get a strong hint to use a double-angle identity. Work with the left side and substitute for $\sin 2x$.

$$\frac{\sin x}{\sin 2x} = \frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x} \quad \text{(double-angle identity)}$$
$$= \frac{1}{2\cos x}$$
$$= \frac{\sec x}{2} \quad \text{(reciprocal identity)}$$

Therefore, LS = RS.

Example 2

Prove that $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

Solution

Work with the left side of the equation.

$$\frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x \cos x}{1-\left(1-2\sin^2 x\right)} \quad \text{(double-angle identities)}$$

$$= \frac{2\sin x \cos x}{2\sin^2 x}$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x \quad \text{(reciprocal identity)}$$

Therefore, LS = RS.

Example 3

Show that $\dfrac{\cos 2x + \cos x}{\sin 2x - \sin x} = \dfrac{\cos x + 1}{\sin x}$.

Solution

Work with the left side of the equation.

$$\frac{\cos 2x + \cos x}{\sin 2x - \sin x} = \frac{2\cos^2 x - 1 + \cos x}{2\sin x \cos x - \sin x} \quad \text{(double-angle identities)}$$

$$= \frac{2\cos^2 x + \cos x - 1}{2\sin x \cos x - \sin x} \quad \text{(rearrange order)}$$

$$= \frac{\cancel{(2\cos x - 1)}(\cos x + 1)}{\sin x\cancel{(2\cos x - 1)}} \quad \text{(Factor numerator and denominator.)}$$

$$= \frac{\cos x + 1}{\sin x}$$

Therefore, LS = RS.

1. Prove the following identities using the double-angle identities and any of the others as necessary.

   a. $(\sin x + \cos x)^2 = 1 + \sin 2x$

   b. $\sec 2x(\cos x - \sin x) = \dfrac{1}{\cos x + \sin x}$

   c. $\dfrac{2}{1-\cos 2x} = \csc^2 x$

2. Derive an identity for $\tan 2a$ using the sum identity for tangent.

Check your answers by turning to the Appendix.

Half-Angle Identities

In addition to double-angles like $\sin 2x$, $\cos 2x$, and $\tan 2x$, you may also encounter half-angles such as $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, and $\tan \frac{x}{2}$ in calculus. Developing identities for these angles will facilitate your work with them.

Half-angle identities are derived from the identities for $\cos 2\theta$. Recall the double-angle identity for cosine from the previous activity.

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= 2\cos^2 \theta - 1$$

$$= 1 - 2\sin^2 \theta$$

If you solve these last two identities for $\cos^2 \theta$ and $\sin^2 \theta$, you get the following:

$$2\cos^2 \theta = 1 + \cos 2\theta \qquad 2\sin^2 \theta = 1 - \cos 2\theta$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta \qquad \sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

To get a half-angle, replace θ with $\frac{x}{2}$,

$$\cos^2 \frac{x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2\left(\frac{x}{2}\right) = \frac{1}{2} + \frac{\cos x}{2} \qquad \sin^2 \frac{x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2\left(\frac{x}{2}\right) = \frac{1}{2} - \frac{\cos x}{2}$$

By taking the square root of both sides, you get a pair of half-angle identities.

$$\cos \frac{x}{2} = \pm\sqrt{\frac{1}{2} + \frac{\cos x}{2}} \qquad \sin \frac{x}{2} = \pm\sqrt{\frac{1}{2} - \frac{\cos x}{2}}$$

Note: It is important to remember that these half-angle identities are rearranged forms of the identities for $\cos 2\theta$, with θ replaced by $\frac{x}{2}$.

These half-angle identities can be used to aid in the proofs of other identities.

Example

Using half-angle identities, show that $\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2 = 1 - \sin x$.

Solution

Work with the left side of the equation.

$$\begin{aligned}
&\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2 \\
&= \sin^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2} \\
&= \left(\pm\sqrt{\frac{1}{2} - \frac{\cos x}{2}}\right)^2 - 2\sin \frac{x}{2} \cos \frac{x}{2} + \left(\pm\sqrt{\frac{1}{2} + \frac{\cos x}{2}}\right)^2 \\
&= \left(\frac{1}{2} - \frac{\cos x}{2}\right) - 2\sin \frac{x}{2} \cos \frac{x}{2} + \left(\frac{1}{2} + \frac{\cos x}{2}\right) \\
&= 1 - 2\sin \frac{x}{2} \cos \frac{x}{2} \\
&= 1 - \sin 2\left(\frac{x}{2}\right) \quad \text{(double-angle identity)} \\
&= 1 - \sin x
\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

3. Use the half-angle identities to prove the following equation.

$$\sin^2 \frac{x}{2} \cos^2 \frac{x}{2} = \frac{\sin^2 x}{4}$$

Check your answers by turning to the Appendix.

You will use these formulas again when differentiating trigonometric functions.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

There is a difference between a conditional equation and an identity. A conditional equation is a true equation for only some specific value(s) of the variable(s), while an identity is true for any value(s) of the variable(s).

Conditional Equation: $x + 4 = 9$

Identity: $x - 2 = 4x - 5 - 3x + 3$

The conditional equation can be true only when $x = 5$, whereas any value of x will transform the identity into a true equation.

Some algebraic expressions contain combinations of trigonometric functions, and the solution will often depend on your ability to change the form of one or both of these expressions so that they are more useful.

It is not always obvious that both sides of a trigonometric equation are equal; thus, a proof is required. A proof involves showing that the left side of the equation is identical to the right side. These sentences are called identities. Proving identities containing trigonometric functions is a form of problem solving.

There are eight fundamental identities that are used to transform trigonometric expressions into equivalent forms.

- $\sin \theta = \dfrac{1}{\csc \theta}$
- $\cos \theta = \dfrac{1}{\sec \theta}$
- $\tan \theta = \dfrac{1}{\cot \theta}$
- $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$
- $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$
- $\sin^2 \theta + \cos^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $1 + \cot^2 \theta = \csc^2 \theta$

Simplifying and transforming trigonometric expressions usually involves making substitutions from these eight identities. In addition to these eight, some expressions may require the use of the sum and difference, double-angle, or half-angle identities. (These can be found listed in Activities 2 and 3 of this section.) Rearrangement of terms and algebraic manipulations may also be necessary.

The three ways in which an identity can be verified are as follows:

- by changing the left side directly to the right side
- by changing the right side directly to the left side
- by changing both the left and right sides to the same form, and doing each separately and independently of each other

Before proving an identity, remember the following:

- Study the whole identity carefully, and try to discover whether any direct application of the fundamental relations or the sum and difference, double-angle, or half-angle identities would be helpful. (For example, when squares of trigonometric functions are involved, try the Pythagorean identities.)
- If you are unable to determine the particular identity that should be applied, change all trigonometric functions to sines and/or cosines. (For example, change $\tan\theta$ to $\frac{\sin\theta}{\cos\theta}$, $\sec\theta$ to $\frac{1}{\cos\theta}$, and so on.)
- It is advisable to begin with the side of the identity that has the greater number of terms.

When attempting to prove identities, try to anticipate the consequences of a step before you implement it. Is the substitution going to allow you to cancel or combine terms? Will the step express a function in a form that is easier to interpret? Will it bring you closer to your goal of proving the identity?

Example 1

Prove the identity $\csc x - \cos x \cot x = \sin x$.

Solution

Work with the left side of the equation since it is more complicated.

$$\begin{aligned} \csc x - \cos x \cot x &= \frac{1}{\sin x} - \cos x \bullet \frac{\cos x}{\sin x} && \text{(reciprocal and quotient identities)} \\ &= \frac{1-\cos^2 x}{\sin x} \\ &= \frac{\sin^2 x}{\sin x} && \text{(Pythagorean identity)} \\ &= \sin x \end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

Example 2

Prove the identity $\cos^2 x - \sin^2 x = \cos 2x$.

Solution

You may work with either side of the equation. The left side of the equation is shown here.

$$\begin{aligned}
\cos^2 x - \sin^2 x &= \cos^2 x - \left(1 - \cos^2 x\right) && \text{(Pythagorean identity)} \\
&= \cos^2 x - 1 + \cos^2 x \\
&= 2\cos^2 x - 1 \\
&= \cos 2x && \text{(double-angle identity)}
\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

Example 3

Prove the identity $\frac{1}{1+\cos A} + \frac{1}{1-\cos A} = 2\csc^2 A$.

Solution

Work with the left side of the equation since it is more complicated.

$$\begin{aligned}
\frac{1}{1+\cos A} + \frac{1}{1-\cos A} &= \frac{(1-\cos A)+(1+\cos A)}{(1+\cos A)(1-\cos A)} && \text{(common denominator)} \\
&= \frac{2}{1-\cos^2 A} \\
&= \frac{2}{\sin^2 A} && \text{(Pythagorean identity)} \\
&= 2\csc^2 A && \text{(reciprocal identity)}
\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

Example 4

Prove the identity $\frac{\tan x - \sin x}{\tan x \sin x} = \frac{1-\cos x}{\sin x}$.

Solution

Transform the left side of the equation.

$$\frac{\tan x - \sin x}{\tan x \sin x} = \frac{\frac{\sin x}{\cos x} - \sin x}{\frac{\sin x}{\cos x} \sin x} \quad \text{(quotient identity for tangent)}$$

$$= \frac{\frac{\sin x - \sin x \cos x}{\cos x}}{\frac{\sin^2 x}{\cos x}} \quad \text{(common denominator)}$$

$$= \frac{\sin x - \sin x \cos x}{\cancel{\cos x}} \bullet \frac{\cancel{\cos x}}{\sin^2 x}$$

$$= \frac{\sin x(1 - \cos x)}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin x}$$

Therefore, LS = RS.

Try some on your own.

1. State whether the following are true or false.

a. $\sin\theta = \frac{1}{\cos\theta}$

b. $\csc^2\theta - \cot^2\theta = 1$

c. $\cos x + \sin x = 1$

d. $\cot x = \frac{1}{\tan x}$

2. State an equivalent expression for each of the following.

a. $1 - \cos^2\theta$

b. $\tan^2 A$

c. $\frac{1}{\csc^2\beta}$

d. $\frac{\cos^2\alpha}{\sin^2\alpha}$

e. $\csc^2 x - 1$

3. Find a common denominator for each of the following.

a. $\frac{\cos x}{\csc x - 1} - \frac{\sin x}{\csc x + 1}$

b. $\frac{\cot A}{\cot A + \cos A} + \frac{1}{\cos A}$

c. $\frac{2}{\sin^2\theta - \cos^2\theta} - \frac{3}{\sin^2\theta + \sin\theta\cos\theta}$

4. Factor and simplify.

a. $\frac{\sin x + 1}{\sin^2 x - 1}$

b. $\cos^3 A + \cos A \sin^2 A$

c. $\cot^2\beta \tan\beta$

d. $\cos x \csc x - \cos x + \csc x - 1$

Check your answers by turning to the Appendix.

Enrichment

These identities are more unusual. Do them if you enjoy a challenge.

1. Prove the following identities.

a. $\sin(A+B)-\sin(A-B)=2\cos A\ \sin B$

b. $\dfrac{\cos(x-y)}{\cos x\ \sin y}=\cot y+\tan x$

2. By writing $a+b+c$ as $(a+b)+c$, determine formulas, in terms of sines and cosines, for the following:

a. $\sin(a+b+c)$ **b.** $\cos(a+b+c)$

c. $\tan(a+b+c)$

3. Can you use the identities for $\tan(x+y)$ and $\tan(x-y)$ to simplify $\tan\left(\frac{\pi}{2}+\theta\right)$ and $\tan\left(\frac{\pi}{2}-\theta\right)$? Justify your answer.

4. Use your knowledge of identities to prove
$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

5. Using the method from question 4, develop identities for $\cos 3\theta$ and $\tan 3\theta$.

Check your answers by turning to the Appendix.

Conclusion

In this section you extended your knowledge of the fundamental trigonometric identities. You expanded the sum and difference identities for sine and cosine to derive sum and difference identities for tangent. Also, you used your knowledge of these sum and difference identities to develop the double-angle identities. You then extended the sum and difference identities for sine, cosine, and tangent to derive half-angle identities for the three primary trigonometric ratios.

All of these identities were then put to use in proving that two trigonometric expressions were equivalent. By transforming one or both sides of an equality, you were able to get equivalent expressions. The left and right sides of a trigonometric identity are equivalent forms, just like you and your mirror image. Will you ever be able to look in a mirror again without being reminded of trigonometric identities?

You will use some of these identities in Section 2 to help you solve trigonometric equations.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

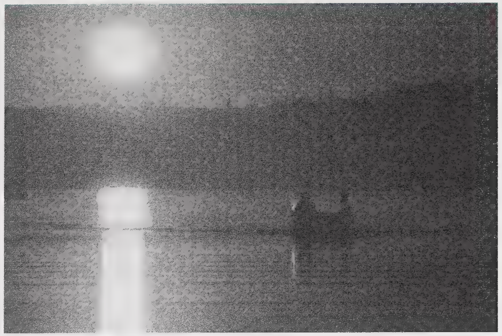
Section 2: Trigonometric Equations

There seems to be a certain mystique attached to a ride on a Ferris wheel. Because of this, it remains, after many years, a very popular ride at any exhibition, midway, or country fair. Young and old alike are thrilled by the feelings that its height and rotation produce. Do you think George Ferris ever dreamed his invention would still be so popular?

Have you ever considered the path that you, as a passenger on a Ferris wheel, follow as you ride? If you charted your path, you would find that it provides an example of a periodic phenomena that can be described using trigonometry. (Thinking of it in these terms is a sure way to remove any excitement produced by the experience!) You could determine your height above the ground at any moment during the trip if you knew the diameter of the wheel and the number of chairs on the wheel. To determine this, you would have to solve a trigonometric equation.

The skills you learn in mathematics are used over and over. Many times you have used the principles of solving equations and applied them to problem solving. In this section you will apply those equation-solving principles to trigonometry.

Activity 1: Specific Solutions



The time of day the sun sets varies during the course of the year. In northern Alberta, sunset times t, measured on the 24-hour clock, may be approximated by $t = -2.5 \cos \frac{2\pi}{365}(d+10) + 17.7$, where d is the day of the year numbered from January 1.

To determine the days of the year when the sunset occurs at 20:00 h (or 8:00 P.M.), you would solve the following equation:

$$20 = -2.5 \cos \frac{2\pi}{365}(d+10) + 17.7$$

An equation involving one or more trigonometric functions of a variable is called a **trigonometric equation**. Some examples are as follows:

- $\sin x = 1$
- $2 \cos 2x - \sin x = 0$
- $\tan x - x = 0$

These are not trigonometric equations:

- $2y + \cos 2 = 4$
- $\sin \frac{\pi}{2} - 3 = 0$
- $\cos 2.5 = x^2 + \sin 3$

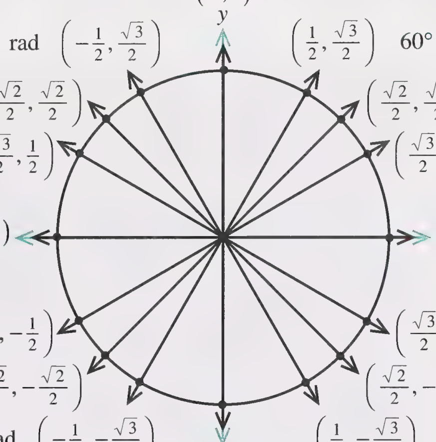
In algebraic equations, the unknown is expressed as a variable and the solution is a real or complex number. In trigonometric equations, the unknown is expressed as a trigonometric function and the solution is the value of an angle in degree or radian measure. (The angles in this section will be expressed in terms of exact radian measure whenever possible, and in approximate radian measure when a calculator is used.)

An example of an algebraic equation is $2x + 1 = 0$.

A similar trigonometric equation might be $2 \sin \theta + 1 = 0$.

In an algebraic equation, you are to find the value of x that satisfies the condition. However, in a trigonometric equation, you must find the value of the angle(s) θ (not $\sin \theta$) that satisfies the condition. This requires an extra step. You will use identities and algebraic manipulations to transform the equation into one or more equations containing a single trigonometric function. After solving for the trigonometric function, you will use the properties of these functions, graphs, the unit circle, or a calculator to find the angle.

Recall that the unit circle was developed in Mathematics 30 and is as follows:



Remember that the ordered pair for each angle represents the cosine and sine values for that angle. For example, look at the angle $\frac{\pi}{6}$:

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ means } \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}.$$

Since trigonometric functions are periodic, a trigonometric equation usually has an infinite number of solutions. If the roots in a specific domain are required, that domain is specified following the equation. For example, $0 \le \theta \le 2\pi$ specifies that only values of θ, measured in radians, within one rotation will be considered solutions.

In general, the values of trigonometric functions are transcendental numbers and can be approximated to any desired accuracy using infinite series.

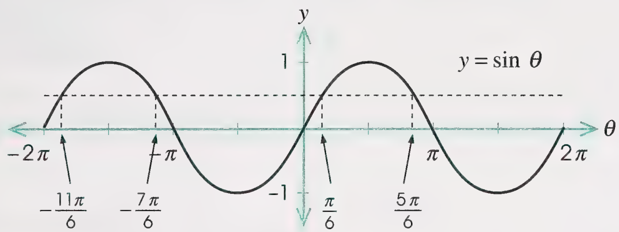
Example 1

Solve $\sin\theta = \frac{1}{2}$, where $-2\pi \le \theta \le 2\pi$.

Solution

You must find all the values of θ, between -2π and 2π, where sine is $\frac{1}{2}$.

Method 1: Using the Graph of $y = \sin\theta$



From the graph, $y = \sin\theta = \frac{1}{2}$ when $\theta = -\frac{11\pi}{6}$, $\theta = -\frac{7\pi}{6}$, $\theta = \frac{\pi}{6}$, and $\theta = \frac{5\pi}{6}$.

Therefore the solutions are $-\frac{11\pi}{6}$, $-\frac{7\pi}{6}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$.

Method 2: Using a Calculator

Note: The calculator should be in radian mode.

You are given the value of the trigonometric function. To solve for θ, use a calculator and key in the following:

0 • 5 INV sin

0.523598775

$$\therefore\ \theta \doteq 0.523\,598\,775$$
$$= \frac{\pi}{6}$$

Note: The calculator you are using may require the key strokes in a different order. Refer to the manual.

To find the other angles that also have a sine of 0.5, you must use the property of sines that you learned in Mathematics 30. If θ_1 is one root, then another root is $\theta_2 = \pi - \theta_1$ (in radians). All other roots can be found by adding or subtracting multiples of 2π to θ_1 or θ_2.

Therefore, if $\theta_1 = \frac{\pi}{6}$, then $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

And, the negative angles are found by $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ and $\frac{5\pi}{6} - 2\pi = -\frac{7\pi}{6}$.

Thus, the solutions are $-\frac{11\pi}{6}$, $-\frac{7\pi}{6}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$.

You may need to use your factoring skills to solve equations.

Example 2

Solve $\cos^2 x - \cos x = 0$, where $0 \le x \le 2\pi$.

Solution

$$\cos^2 x - \cos x = 0$$
$$\cos x(\cos x - 1) = 0$$

Think of $\cos^2 x$ as $(\cos x)^2$ in order to factor.

$$\cos x = 0 \quad \text{or} \quad \cos x - 1 = 0$$
$$x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \qquad \cos x = 1$$
$$x = 0 \text{ and } 2\pi$$

The solutions are 0, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and 2π.

Example 3

Find the values of x if $2\sin^2 x - 7\sin x + 3 = 0$, where $0 \le x \le 2\pi$.

Solution

$$2\sin^2 x - 7\sin x + 3 = 0$$
$$(2\sin x - 1)(\sin x - 3) = 0$$

Factor as you would a trinomial.

$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x - 3 = 0$$
$$\sin x = \frac{1}{2} \qquad \sin x = 3$$
$$x = \frac{\pi}{6} \text{ and } \frac{5\pi}{6}$$

Since the maximum value of $\sin x$ is 1, there are no values of x such that $\sin x = 3$; thus, this part of the equation is invalid. Therefore, the solutions are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Example 4

Solve $\sin^2 x = 1$, where $-\pi < x < 2\pi$.

Solution

$$\sin^2 x = 1$$
$$\sin^2 x - 1 = 0 \quad \text{(Rearrange.)}$$
$$(\sin x - 1)(\sin x + 1) = 0 \quad \text{(Factor.)}$$

$$\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$
$$\sin x = 1 \qquad \sin x = -1$$
$$x = \frac{\pi}{2} \qquad x = -\frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

Thus, the solutions are $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$.

1. Find the solutions for the following equations for the given intervals.

   a. $\sqrt{2}\sin\theta = 1$, where $-2\pi < \theta < 2\pi$
   b. $\tan^2 x = 3$, where $0 \le x \le 2\pi$
   c. $\sin x(\cos x - 1) = 0$, where $0 \le x \le 2\pi$
   d. $2\cos^2\theta + 5\cos\theta - 3 = 0$, where $-2\pi < \theta < 2\pi$
   e. $\sin^2 x = \sin x + 2$, where $-\pi < x < \pi$

Check your answers by turning to the Appendix.

The following example shows that you may have to use the identities that you studied in the first section to put the equation into a form that is solvable.

Example 5

Find the solutions for $\cos^2 x + 2\sin x - 2 = 0$, where $0 \le x \le 2\pi$.

Solution

First, transform the equation so that only one trigonometric function is used. Remember that the Pythagorean identities say that $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}
\cos^2 x + 2\sin x - 2 &= 0 \\
\left(1 - \sin^2 x\right) + 2\sin x - 2 &= 0 \\
\sin^2 x - 2\sin x + 1 &= 0 \quad \text{(rearranged and simplified)} \\
(\sin x - 1)(\sin x - 1) &= 0 \\
\sin x - 1 &= 0 \\
\sin x &= 1 \\
x &= \frac{\pi}{2}
\end{aligned}$$

The solution is $\frac{\pi}{2}$.

Sometimes the solutions to trigonometric equations are not as readily found. You may have to use the quadratic formula and then a calculator to find the measures of the angles that satisfy the equation. The angles you find will be approximate radian measures.

Example 6

Solve the equation $2\cos x - 1 = 2\sec x$, where $0 \le x \le 2\pi$.

Solution

Use a trigonometric identity to change the equation to one that involves only the cosine function.

$$\begin{aligned}
2\cos x - 1 &= 2\sec x \\
2\cos x - 1 &= \frac{2}{\cos x}
\end{aligned}$$

Multiply each term by $\cos x$ to remove the $\cos x$ from the denominator.

$$2\cos^2 x - \cos x = 2$$
$$2\cos^2 x - \cos x - 2 = 0$$

This quadratic cannot be factored. Use the quadratic formula to find its solutions.

$$\begin{aligned}\cos x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{1 \pm \sqrt{17}}{4}\end{aligned}$$

$$\cos x = \frac{1-\sqrt{17}}{4} \quad \text{or} \quad \cos x = \frac{1+\sqrt{17}}{4}$$

$$\cos x \doteq -0.780\,776\,407$$
$$x \doteq 2.47$$

There is no solution for this part of the equation.

This angle is in the second quadrant. Since cosine is also negative in the third quadrant, a second angle is $x_2 \doteq 2\pi - 2.47 \doteq 3.81$ rad.

The solutions are angles that measure approximately 2.47 rad and 3.81 rad.

2. Use the skills for solving equations to find the solution for these trigonometric equations.

   a. $\cos^2 x = 2 - 2\sin x$, where $0 \le x \le 2\pi$

   b. $\dfrac{\cos x + 2}{\sec x} = 3$, where $0 \le x \le 2\pi$

   c. $\sin x = \cos x$, where $0 \le x \le 2\pi$

   d. $\sec x = 2\cos x + 2$, where $0 \le x \le 2\pi$

Check your answers by turning to the Appendix.

Equations that contain functions of $n\theta$ angles require special attention. For example, there are two angles in the interval $[0, 2\pi]$ whose sine is $\frac{1}{2}$. Sine is positive in the first and second quadrants, and the angles are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus, the equation $\sin\theta = \frac{1}{2}$, where $0 \le x \le 2\pi$, has both $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ as solutions. The problem is more complex for functions of $n\theta$.

Example 7

Solve $\sin 2\theta = \frac{1}{2}$, $0 \le \theta \le 2\pi$.

Solution

In this example, the condition is 2θ. Since θ can be any angle between 0 and 2π, then 2θ can be any angle between 0 and $2(2\pi)$ or 4π. Thus, 2θ can be $\frac{\pi}{6}$ plus 2π (once around) or $\frac{13\pi}{6}$.

$$\therefore\ 2\theta = \frac{\pi}{6} \quad \text{and} \quad 2\theta = \frac{13\pi}{6}$$

$$\theta = \frac{\pi}{12} \qquad \theta = \frac{13\pi}{12}$$

Divide by 2.

Also, since 2θ can equal $\frac{5\pi}{6}$, 2θ can equal $\frac{5\pi}{6}$ plus 2π or $\frac{17\pi}{6}$.

$$\therefore\ 2\theta = \frac{5\pi}{6} \quad \text{and} \quad 2\theta = \frac{17\pi}{6}$$

$$\theta = \frac{5\pi}{12} \qquad \theta = \frac{17\pi}{12}$$

The equation $\sin 2\theta = \frac{1}{2}$ has solutions $\frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{13\pi}{12}$, and $\frac{17\pi}{12}$.

From Example 7, a pattern develops. If $\sin \theta = c$ has two solutions, and $\sin 2\theta = c$ has four solutions, then $\sin 3\theta = c$ will have six solutions, $\sin 4\theta = c$ will have eight solutions, and $\sin n\theta = c$ will have $2n$ solutions.

Example 8

Solve $\cos 3x = -\frac{\sqrt{2}}{2}$, where $0 \le x \le 2\pi$.

Solution

If $\cos x = -\frac{\sqrt{2}}{2}$, the solutions would be $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

Since the condition is $\cos 3x$, the solution could be any angle between 0 and $3(2\pi)$ or 6π.

$$\frac{3\pi}{4} + 2\pi \text{ (once around)} = \frac{11\pi}{4}$$

$$\frac{3\pi}{4} + 4\pi \text{ (twice around)} = \frac{19\pi}{4}$$

$$\frac{5\pi}{4} + 2\pi \text{ (once around)} = \frac{13\pi}{4}$$

$$\frac{5\pi}{4} + 4\pi \text{ (twice around)} = \frac{21\pi}{4}$$

$$\therefore\ 3x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \text{ and } \frac{21\pi}{4}$$

$$x = \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \text{ and } \frac{21\pi}{12}$$

Divide by 3.

The equation $\cos 3x = -\frac{\sqrt{2}}{2}$ has the six solutions: $\frac{\pi}{4}$, $\frac{5\pi}{12}$, $\frac{11\pi}{12}$, $\frac{13\pi}{12}$, $\frac{19\pi}{12}$, and $\frac{7\pi}{4}$.

Example 9

Solve for x if $\tan 4x - \frac{\sqrt{3}}{3} = 0$, where $0 < x < 2\pi$.

Solution

The interval given with the equation indicates that angles between 0 and 2π are acceptable solutions to the equation. First, solve for $\tan 4x$.

$$\tan 4x = \frac{\sqrt{3}}{3} \text{ or } \frac{1}{\sqrt{3}}$$

Recall that $\tan = \frac{\sin}{\cos}$.

There are two angles between 0 and 2π whose tangent is $\frac{1}{\sqrt{3}}$. They are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.

$$\therefore \; 4x = \frac{\pi}{6} \text{ and } \frac{7\pi}{6}$$

Since x must be between 0 and 2π, $4x$ must be between 0 and $4(2\pi) = 8\pi$. That means $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ can be increased by 2π (period), 4π (two periods), and 6π (three periods).

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \qquad \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$$

$$\frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \qquad \frac{7\pi}{6} + 4\pi = \frac{31\pi}{6}$$

$$\frac{\pi}{6} + 6\pi = \frac{37\pi}{6} \qquad \frac{7\pi}{6} + 6\pi = \frac{43\pi}{6}$$

$$\therefore \; 4x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}, \frac{37\pi}{6}, \text{ and } \frac{43\pi}{6}$$

$$x = \frac{\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{19\pi}{24}, \frac{25\pi}{24}, \frac{31\pi}{24}, \frac{37\pi}{24}, \text{ and } \frac{43\pi}{24}$$

Therefore, the eight solutions to the equation are $\frac{\pi}{24}$, $\frac{7\pi}{24}$, $\frac{13\pi}{24}$, $\frac{19\pi}{24}$, $\frac{25\pi}{24}$, $\frac{31\pi}{24}$, $\frac{37\pi}{24}$, and $\frac{43\pi}{24}$.

3. Solve the following multiple-angle equations for the interval $[0, 2\pi]$.

   a. $\sin 3x = -\frac{1}{2}$

   b. $\tan 2x + 1 = 0$

   c. $\cos 4x - 1 = 0$

   d. $2 \sin x \cos x = 0$

Check your answers by turning to the Appendix.

You may wish to solve the sunset problem described earlier in this activity. The sun sets at 20:00 h (or 8:00 P.M.) on May 30 and August 14.

Activity 2: General Solutions

Because of the periodic nature of trigonometric functions, if a specific interval for the solutions of a trigonometric equation is not given, then there are an infinite number of solutions possible. The examples in Activity 1 all had a specified domain in which the solutions were to be found. You will now look at some examples where a domain is not specified; so you must state general solutions.

For example, consider the equation $\sin \theta = 0$.

You know that 0, π, and 2π are solutions within one rotation of the unit circle. But these are not the only solutions. Any one of the following values of θ are solutions.

$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

This infinite solution set can be written as $\{n\pi, n \text{ is an integer}\}$.

Example 1

Solve for θ if $\sin \theta = -\frac{\sqrt{3}}{2}$.

Solution

To solve this equation, two observations need to be made:

- Sine is negative in the third and fourth quadrants.
- Sine is $-\frac{\sqrt{3}}{2}$ when $\theta = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

These solutions exist in the interval $[0, 2\pi]$. To list the general solutions, addressing the periodic nature of trigonometric functions, add $2n\pi$ (signifying consecutive wraps of the unit circle) to each angle.

Therefore, the general solutions to $\sin \theta = -\frac{\sqrt{3}}{2}$ are $\frac{4\pi}{3} + 2n\pi$ and $\frac{5\pi}{3} + 2n\pi$, where n is an integer.

Example 2

Find x if $\tan^2 x - \tan x = 0$.

Solution

$$\tan^2 x - \tan x = 0$$
$$\tan x(\tan x - 1) = 0$$

For the interval $[0, 2\pi]$ the solutions are as follows:

$$\tan x = 0 \qquad \text{or} \qquad \tan x - 1 = 0$$
$$x = 0, \pi, \text{ and } 2\pi \qquad \tan x = 1$$
$$x = \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$$

The general solutions would be $0 + 2n\pi$ or $2n\pi$, $\frac{\pi}{4} + 2n\pi$, $\pi + 2n\pi$, and $\frac{5\pi}{4} + 2n\pi$, where n is an integer. These may be combined as $n\pi$ and $\frac{\pi}{4} + n\pi$ where n is an integer.

To find the general solutions for any trigonometric equation, solve for the angle in the interval $[0, 2\pi]$. For each solution in this initial period, add $2n\pi$ to obtain general solutions.

Remember: In order to solve trigonometric equations, you may use the identities and algebraic manipulations to transform the equation into equations that contain a single trigonometric function (whose solution can be read from the unit circle or a calculator).

Find the general solutions for the following equations.

1. $2\cos^2 x - 1 = 0$

2. $2\sin^2 \theta - \sin\theta = 1$

3. $\sec x \csc x = 2\csc x$

4. $\cos^2 x + \sin x = 1$

5. $\cos\frac{x}{2} - \cos x = 1$

Check your answers by turning to the Appendix.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Although a variety of methods can be used to solve trigonometric equations, the following strategies are most often applied.

- Use formulas and identities already in place to transform the equation to terms of a single trigonometric function.
- Use the double- or half-angle identities whenever necessary.
- Use algebraic manipulations to isolate the trigonometric function, such as adding, subtracting, multiplying, dividing, factoring, and/or finding square roots.
- Solve for the angle by reading exact values from the unit circle if possible, or use a calculator to find approximate values for the angle or angles that satisfy the condition.
- In the case of equations containing multiple angles $(n\theta)$, divide each angle measure by n to find solutions.
- To find general solutions for trigonometric equations, find the specific solutions in the interval $[0, 2\pi]$ and add $2n\pi$ to each angle.

Example 1

Find θ, such that $2 \sin \theta - 1 = 0$, where $0 \le \theta \le 2\pi$.

Solution

First, notice that there is a restriction on the domain, indicating that the value of the angle must be between 0 and 2π inclusive.

Isolate the trigonometric function by using algebraic methods.

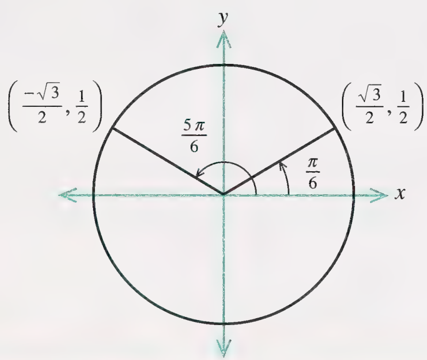
$$\begin{aligned} 2 \sin \theta - 1 &= 0 && \text{(Add 1 to both sides.)} \\ 2 \sin \theta &= 1 && \text{(Divide both sides by 2.)} \\ \sin \theta &= \frac{1}{2} && \text{(Now } \sin \theta \text{ is isolated.)} \end{aligned}$$

Now recognize the function in terms of a ratio and determine a possible value of the angle θ from the unit circle.

By examining the unit circle notice that

$\sin \theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$

and $\frac{5\pi}{6}$.



Therefore, if $2 \sin \theta - 1 = 0$, where $0 \le \theta \le 2\pi$, then $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Example 2

Solve $\cos^2 x - 1 = 0$.

Solution

Notice that there is no interval specified for the solutions. Therefore, you must list the general solutions of $\cos^2 x - 1 = 0$.

There are two ways to isolate the trigonometric function:

Method 1

Factor the expression as you would for an algebraic equation.

$$\cos^2 x - 1 = 0$$
$$(\cos x - 1)(\cos x + 1) = 0$$
$$\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$\cos x = 1 \qquad\qquad \cos x = -1$$

Method 2

Add +1 to both sides of the equation and then take the square root of both sides.

$$\begin{aligned} \cos^2 x - 1 &= 0 \\ \cos^2 x &= 1 \\ \cos x &= \pm 1 \end{aligned}$$

Note that both methods bring you to the same conclusion. Your task now is to find the value(s) of x that satisfy the condition.

Examining the unit circle, you find that cosine is ± 1 when $x = 0$, π, and 2π.

Thus, the general solution is $n\pi$, where n is an integer.

You may need to use a calculator to solve an equation if an exact value does not exist.

Example 3

Solve for x if $-4(\sin x - 1) - 3 = 0$, where $0 \le x \le 2\pi$.

Solution

$$-4(\sin x - 1) - 3 = 0$$
$$-4(\sin x - 1) = 3$$
$$\sin x - 1 = -\frac{3}{4}$$
$$\sin x = -\frac{3}{4} + 1$$
$$\sin x = \frac{1}{4} \text{ or } 0.25$$

There is no angle on the unit circle for which sine is 0.25; therefore, you must use a calculator to determine an approximate value for the angle.

0 • 2 5 INV sin

0.252680255

Therefore, $x \doteq 0.25$ rad.

Remember that sine is positive in the first and second quadrants. 0.25 rad is an angle in the first quadrant. The angle in the second quadrant is determined as follows:

π − 0 • 2 5 INV sin =

2.888912398

Therefore, $x \doteq 2.89$ rad.

The angle measuring 2.89 rad is found in the second quadrant.

The solutions for $-4(\sin x - 1) - 3 = 0$, when $0 \le x \le 2\pi$ are approximately 0.25 rad and 2.89 rad.

Sometimes you must factor to isolate the variable.

Example 4

Solve $6\sin^2\theta - 5\sin\theta + 4 = 0$, for the interval $[-\pi, \pi]$.

Solution

Substitute x for $\sin\theta$.

$$6x^2 - 5x + 4 = 0$$

$$(2x+1)(3x-4) = 0 \quad \text{(Factor.)}$$

$$2x + 1 = 0 \quad \text{or} \quad 3x - 4 = 0$$

$$x = -\frac{1}{2} \qquad x = \frac{4}{3}$$

Replace x with $\sin\theta$.

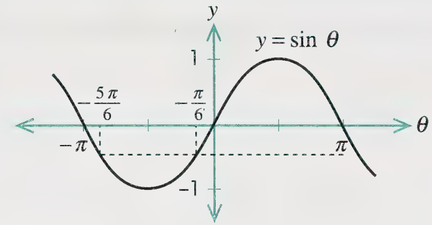
$$\sin\theta = -\frac{1}{2} \quad \text{or} \quad \sin\theta = \frac{4}{3}$$

From the unit circle, for the interval $[0, 2\pi]$, $\sin\theta = -\frac{1}{2}$ when $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

There is no value for sine greater than 1; thus, so $\sin\theta = \frac{4}{3}$ has no solution.

For the interval $[-\pi, \pi]$, the two values of θ become $-\frac{5\pi}{6}$ and $-\frac{\pi}{6}$.

You may check these solutions on the sine graph.



Therefore, the solutions are $-\frac{5\pi}{6}$ and $-\frac{\pi}{6}$.

If more than one trigonometric function or a reciprocal function appears in an equation, use the identities to express it in terms of one function.

Example 5

Solve for x if $\sin^2 x = \cos^2 x + 2\cos x$, where $0 \le x \le 2\pi$.

Solution

Since two trigonometric functions appear in the equation, and cosine seems to be the dominant one, you can change $\sin^2 x$ to cosine by using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

$$1 - \cos^2 x = \cos^2 x + 2\cos x$$

$$2\cos^2 x + 2\cos x - 1 = 0$$

Since this trinomial cannot be factored, use the quadratic formula to solve for $\cos x$.

$$\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{-2 \pm \sqrt{12}}{4}$$
$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$\cos x = \frac{-1+\sqrt{3}}{2} \quad \text{or} \quad \cos x = \frac{-1-\sqrt{3}}{2}$$

$$\cos x = 1.366\,025\,404$$
$$x \doteq 1.2 \text{ rad}$$

There is no solution for this part of the equation.

This angle is in the first quadrant. Another angle which has the same cosine value can be found in the fourth quadrant.

$$x_2 \doteq 2\pi - 1.2$$
$$\doteq 5.1 \text{ rad}$$

Therefore, the two solutions are approximately 1.2 rad and 5.1 rad.

It is possible that you will work with equations containing trigonometric functions of multiple angles like 2θ, 3θ, and so on. These multiple angles $(n\theta)$ affect the domain by a factor of n.

Example 6

Solve $2 \sin 2\theta + 1 = 0$, where $0 \le \theta \le 2\pi$.

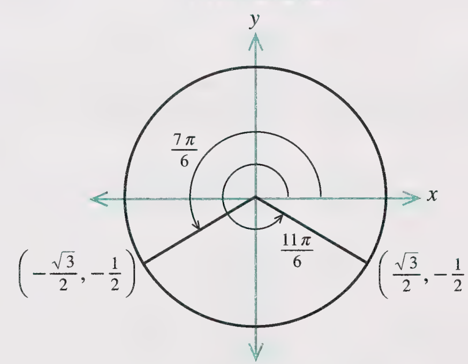
Solution

$$2 \sin 2\theta + 1 = 0$$
$$\sin 2\theta = -\frac{1}{2}$$

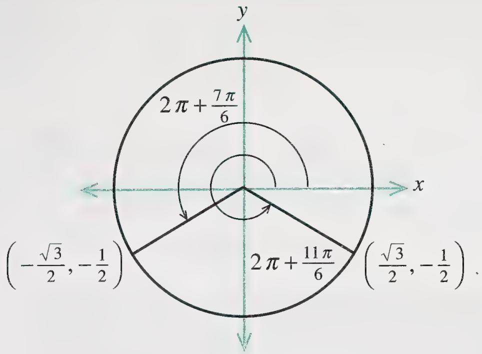
The angle 2θ makes the domain of the solution twice as large; therefore, 2θ will be an angle or angles between 0 and $2(2\pi)$ or 4π.

On the unit circle, in one rotation and two rotations, you will find sine to be $-\frac{1}{2}$ for $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$, and $\frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}$.

First Rotation



Second Rotation



$$\therefore\ 2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \text{ and } \frac{23\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \text{ and } \frac{23\pi}{12}$$

Divide by 2.

Note: These angles still fall within the original domain of $0 \le \theta \le 2\pi$.

The solutions for $2 \sin 2\theta + 1 = 0$ are $\frac{7\pi}{12}$, $\frac{11\pi}{12}$, $\frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

Now try some of these on your own.

1. Solve each trigonometric equation.

a. $2 \sin x - \sqrt{3} = 0$, where $0 \le x \le \pi$

b. $3 \cot x + 3 = 0$, where $0 \le x \le 2\pi$

Use a calculator to answer question 2.

2. Solve the trigonometric equation $-5(\sin x + 1) + 3 = 0$, where $[\pi, 2\pi]$.

3. Solve the following for the unknown.

a. $4 \cos^2 \theta - 3 = 0$

b. $\sin x \cos x - \sin x = 0$, where $\frac{\pi}{2} \le x \le 2\pi$

c. $2 \sin^2 \theta - 1 = \cos \theta$, where $-2\pi \le \theta \le 2\pi$

d. $\sqrt{2} \sin 2\theta = 1$, where $0 \le \theta \le 2\pi$

e. $4 \sin^2 3x - 1 = 0$, where $0 \le x \le 2\pi$

Check your answers by turning to the Appendix.

Enrichment

Scientists and engineers are often faced with problems involving trigonometry, and must have some method to relate these problems mathematically and then solve for the unknown. They use trigonometric functions to model periodic phenomena.

Example 1

A sine or cosine curve approximates the periodic rise and fall of water due to tides. The depth of the water, in a tidal region, as a function of time is expressed by a trigonometric equation.

Some of the highest tides in the world occur in the Bay of Fundy, between Nova Scotia and New Brunswick. It has been determined that the equation to calculate the depth of water at any time during the day is as follows:

$$h = 3\cos\left(2\pi\frac{(t-4.5)}{12.4}\right)+5$$

h is the depth of the water (in metres);
t is the time (in hours) on a 24-hour clock;
4.5 h is the time of the first high tide;
5 m is the mean water level;
12.4 h is the period between two high tides; and
3 m is the difference between high tide and mean tide (amplitude).

Calculate the depth of water (to the nearest tenth) at 7:15 A.M. and 4:30 P.M.

Solution

First, you must convert these times to decimals of hours on the 24-hour clock.

$$7:15 \text{ A.M.} = \left(7+\frac{15}{60}\right) = 07.25 \text{ h} \quad \text{and}$$

$$4:30 \text{ P.M.} = \left(12+4+\frac{30}{60}\right) = 16.5 \text{ h}$$

When $t = 7.25$, $h = 3\cos\left(2\pi\frac{7.25-4.5}{12.4}\right)+5$

Use a calculator in radian mode to find h.

3 × (2 × π × (7 • 2 5 − 4 • 5) ÷ 1 2 • 4) cos + 5 =

5.529259299

Therefore, $h \doteq 5.529\,259\,299$.

At 7:15 A.M. the water is approximately 5.5 m deep.

When $t = 16.5$, $h = 3\cos\left[2\pi\dfrac{(16.5-4.5)}{12.4}\right] + 5$.

Following the same calculator sequence, replacing 7.25 with 16.5, find h.

Therefore, $h \doteq 7.938\,589\,824$.

The depth of the water at 4:15 P.M. is approximately 7.9 m.

This equation also allows you to calculate the time at which the water is a certain depth. You must remember that since this is a periodic function, a particular depth will occur at different times of the day. The water will reach this depth twice within one period and since the period of the function is 12.4 h, it will be this depth four times in 24 h.

Example 2

Using the equation from Example 1, find the times of the day (to the nearest minute) when the water is 7 m deep.

Solution

$$3\cos\left[2\pi\frac{(t-4.5)}{12.4}\right] + 5 = 7$$

$$3\cos\left[2\pi\frac{(t-4.5)}{12.4}\right] = 2$$

$$\cos\left[2\pi\frac{(t-4.5)}{12.4}\right] = \frac{2}{3}$$

To solve this equation, you must find a value whose cosine is $\frac{2}{3}$.

(2 ÷ 3) INV cos

0.841068671

$$\begin{aligned} \therefore\ 2\pi\frac{(t-4.5)}{12.4} &\doteq 0.841\,068\,67 \\ 2\pi(t-4.5) &\doteq 10.429\,251\,52 \\ t-4.5 &\doteq 1.659\,866\,931 \\ t &\doteq 6.159\,866\,931 \text{ h} \end{aligned}$$

Convert the decimal values to minutes.

$$0.159\,866\,931 \times 60 \doteq 9.592\,015\,86 \text{ min}$$

Therefore, the water is 7 m deep at approximately 6:10 A.M.

The second time, in the 12.4 h period, that the water reaches a depth of 7 m is found by taking the difference between the first time and the time of the first high tide, then subtracting this value from the first high tide.

$$6.159\,866\,931 - 4.5 \doteq 1.659\,866\,931$$
$$4.5 - 1.659\,866\,931 \doteq 2.840\,133\,069 \text{ h}$$

Convert the decimal values to minutes.

$$0.840\,133\,069 \times 60 \doteq 50.407\,984\,14 \text{ min}$$

The water is 7 m deep at approximately 2:50 A.M.

To find the times in the second period of the day, add 12.4 h to each of the times.

$6.159\,866\,931 + 12.4 = 18.559\,866\,931$ which is 18:34 or 6:34 P.M.

and

$2.840\,133\,069 + 12.4 = 15.240\,133\,069$ which is 15:14 or 3:14 P.M.

Thus, the water has a depth of 7 m at 2:50 A.M., 6:10 A.M., 3:14 P.M., and 6:34 P.M.

1. At a seaside marina, the equation for the depth of water as a function of time is as follows:

$$h = 2\cos\left(2\pi\frac{(t-5.2)}{12.4}\right) + 6$$

a. Find the depth of the water at 3:00 P.M.
b. Determine one time of the day when the water is 7 m deep.

2. The time the sun sets in Prince Albert, Saskatchewan, is approximated by $t = -2.5 \cos \frac{2\pi}{365}(d+10) + 18.5$, where t is given by the 24-hour clock and d is the day of the year numbered from January 1. When does the sun set at 8:30 P.M.?

Check your answers by turning to the Appendix.

Conclusion

In this section you expanded your skills in solving trigonometric equations. All the skills you learned previously in solving algebraic equations were applied to the solution of trigonometric equations. You found solutions within a specified domain and also determined general solutions when no limiting domain was given. Emphasis was given to trigonometric functions of multiple angles, and you studied the effect multiple angles have on the domain and the solutions of an equation.

Your extended familiarity with this branch of mathematics, through the review of trigonometric functions in Sections 1 and 2, will enable you to study new topics in trigonometry. The next two sections take you back to calculus. You will begin the study of the calculus of trigonometry.

You should now be able to solve real-world problems such as how high above the ground you would be at any instant when riding a ferris wheel. This problem and others which involve periodic motion can be described using trigonometric functions. Your future work will involve modelling such motion and applying the techniques of calculus.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: Limits of Trigonometric Functions

It is a thrill for any skier to be the first one down a mountain after a heavy snowfall. To leave a distinctive mark on fresh snow is every avid skier's dream. A nice long run allows you to develop and maintain a smooth, rhythmic flow.

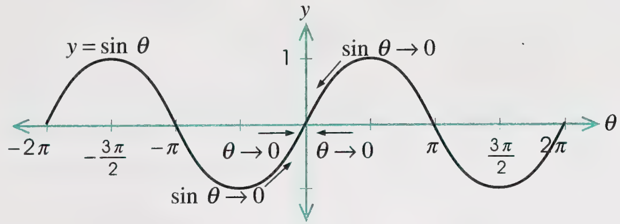
The beautiful S-curves carved in the snow by the skier mimic sinusoidal curves. The pattern created by the skier can be described as a periodic trigonometric function. It would be ideal if you could carve these curves forever; but, alas, there is a limit to your run down the slope—the bottom of the hill. Untracked snow then becomes harder to find as the day progresses. You still create the same curves, but they are lost among all the rest.

In this section you will be studying the limits of trigonometric functions. You studied the limits of algebraic functions in Module 2 and will be using many of the same concepts in your examination of limits in the context of trigonometry. You begin by finding the limits of sine and cosine; you then proceed to more complex trigonometric expressions.

Activity 1: Sine and Cosine

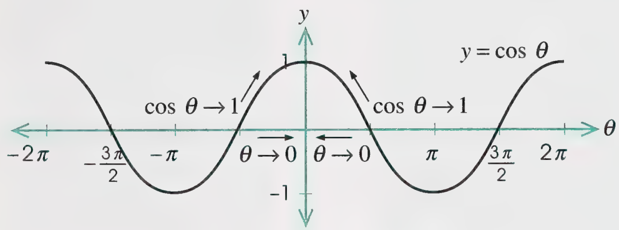
You will begin this section by finding trigonometric limits for simple sine and cosine functions.

First look at the left- and right-hand limits of $\sin\theta$ and $\cos\theta$ by examining their graphs.



$\lim\limits_{\theta \to 0^-} \sin\theta = 0$ $\qquad$ $\lim\limits_{\theta \to 0^+} \sin\theta = 0$

Note: As $\theta \to 0$ from the left or the right, the sine curve is also approaching 0.



$\lim\limits_{\theta \to 0^-} \cos\theta = 1$ $\qquad$ $\lim\limits_{\theta \to 0^+} \cos\theta = 1$

Therefore, $\lim\limits_{\theta \to 0} \sin\theta = 0$ and $\lim\limits_{\theta \to 0} \cos\theta = 1$.

The limits important to trigonometric functions are:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad \text{or} \quad \lim_{\theta \to 0} \frac{\theta}{\sin\theta} \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

These limits are necessary in order to find the derivatives of trigonometric functions—the goal of the next section of the module.

As you learned in Module 2, with algebraic functions, limits can be determined in two ways: numerically and algebraically. You will use a similar idea here when finding the limits of trigonometric functions.

Find $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$.

To examine this limit using a calculator in radian mode, find the value of $\frac{\sin\theta}{\theta}$ for values of θ close to 0 by setting up a table of values. Show the values as θ approaches 0 from the left and from the right.

Approaching 0 from the left

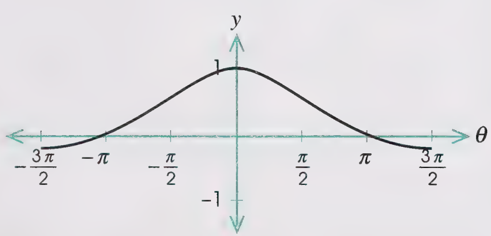
θ	$\frac{\sin \theta}{\theta}$
0.5	0.958 851
0.3	0.985 067
0.1	0.998 334
0.05	0.999 583
0.03	0.999 850
0.01	0.999 983
0.001	0.999 999

Approaching 0 from the right

θ	$\frac{\sin \theta}{\theta}$
−0.5	0.958 851
−0.3	0.985 067
−0.1	0.998 334
−0.05	0.999 583
−0.03	0.999 850
−0.01	0.999 983
−0.001	0.999 999

The tables seem to indicate a trend in the values of $\frac{\sin \theta}{\theta}$. The limit is 1.

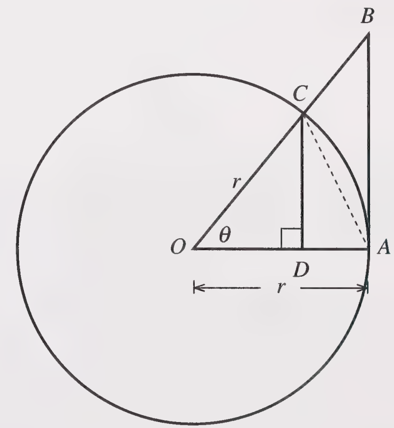
The graph of $y = \frac{\sin \theta}{\theta}$ is:



It also seems to indicate a limit of 1 for $\frac{\sin \theta}{\theta}$ when $\theta \to 0$.

A proof is required. The proof of this limit is shown geometrically. The values of θ will be restricted to $0 < \theta < \frac{\pi}{2}$.

Consider a circle with radius 1.



Since $\overline{OA} = \overline{OC} = 1$, $\sin \theta = \frac{\overline{AB}}{\overline{OB}}$, and $\tan \theta = \frac{\overline{AB}}{\overline{OA}}$, you can say that $\overline{AB} = \overline{OA} \tan \theta = \tan \theta$ and $\overline{CD} = \overline{OC} \sin \theta = \sin \theta$.

Comparing the areas gives you the following:

$$\Delta OAC < \text{sector } OAC < \Delta OAB$$

$$\frac{1}{2} \bullet \overline{OA} \bullet \overline{CD} < \frac{1}{2} r^2 \theta < \frac{1}{2} \bullet \overline{OA} \bullet \overline{AB}$$

> Recall that the area of a sector is found by using the formula $\frac{1}{2} r^2 \theta$.

$$\frac{1}{2}\bullet 1\bullet \sin\theta < \frac{1}{2}\bullet 1\bullet \theta < \frac{1}{2}\bullet 1\bullet \tan\theta \quad \text{(Multiply by 2.)}$$

$$\sin\theta < \theta < \tan\theta$$ (Divide by $\sin\theta$ to put the statement closer to the form you need.)

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ (Invert to get the form you need.)

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Recall that if three positive numbers $2 < 3 < 4$, are replaced by their reciprocals, the inequalities are reversed, $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$ or $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$.

Now, find the limits.

$$\lim_{\theta\to 0}\cos\theta < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < \lim_{\theta\to 0} 1$$

(This is referred to the "squeeze", "pinch", or "sandwich" theorem.)

You know that $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$.

Therefore, for $0 < \theta < \frac{\pi}{2}$, since $\frac{\sin\theta}{\theta}$ is between $\cos\theta$ and 1, it must also approach 1.

The same argument is true for the $\lim_{\theta\to 0}\frac{\theta}{\sin\theta}$.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$$

Most of the limits that you will evaluate in this activity will require you to produce one of these limits, since straight substitution will often produce $\frac{0}{0}$. Some of the methods that you used in Module 2 for evaluating algebraic limits can be used for trigonometric limits. Previously, whenever both the numerator and the denominator of a fraction approached $\frac{0}{0}$, you (after some algebraic manipulation) cancelled a factor to allow the evaluation of the limit. This cannot be done with this limit.

Example 1

Evaluate $\lim_{x\to 0}\frac{\sin x}{2x}$.

Solution

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{2x} &= \lim_{x\to 0}\frac{1}{2}\left(\frac{\sin x}{x}\right)\\ &= \frac{1}{2}\bullet\lim_{x\to 0}\frac{\sin x}{x}\\ &= \frac{1}{2}(1)\end{aligned}$$

Therefore, $\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$.

Example 2

Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$.

Solution

To evaluate $\lim\limits_{x \to 0} \frac{\sin 3x}{x}$, you must compare it to $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

As $x \to 0$, then $3x \to 0$ also.

$$\therefore \lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{3x \to 0} \frac{\sin 3x}{x}$$

Multiply the numerator and the denominator of the expression on the right by 3 to obtain the needed denominator. (You are not changing the value of the expression since you are really only multiplying by 1.)

$$\begin{aligned}
\therefore \lim_{x \to 0} \frac{\sin 3x}{x} &= \lim_{3x \to 0} \frac{3 \sin 3x}{3x} \\
&= 3\left(\lim_{3x \to 0} \frac{\sin 3x}{3x} \right) \\
&= 3(1) \\
&= 3
\end{aligned}$$

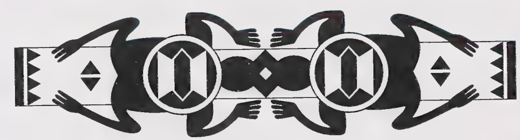
Therefore, $\lim\limits_{x \to 0} \frac{\sin 3x}{x} = 3$.

Example 3

Find $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x}$.

Solution

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin^2 x}{x} &= \lim_{x \to 0} \frac{\sin x \sin x}{x} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \bullet \lim_{x \to 0} \sin x \\
&= 1 \bullet \sin 0 \\
&= 1(0) \\
&= 0
\end{aligned}$$



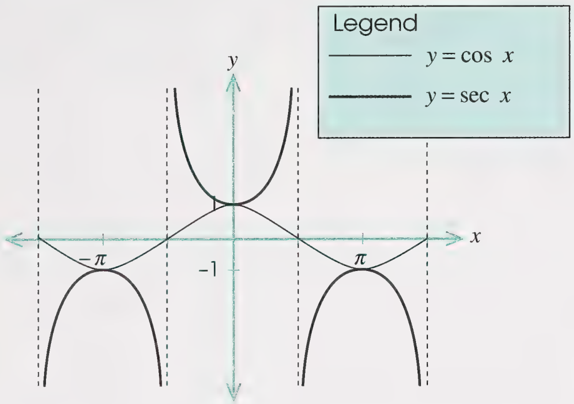
Example 4

Evaluate $\lim\limits_{x \to 0} x \sec x$.

Solution

$$\lim_{x \to 0} x \sec x = \lim_{x \to 0} x \bullet \lim_{x \to 0} \sec x$$

Examine the graph of $y = \sec x$ to find the $\lim \sec x$ as $x \to 0$.



$$\therefore \lim_{x \to 0} x \sec x = \lim_{x \to 0} x \bullet \lim_{x \to 0} \sec x$$
$$= 0(1)$$
$$= 0$$

Example 5

Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x}{\sin x}$.

Solution

$$\begin{aligned}
\lim_{x \to 0} \frac{\tan x}{\sin x} &= \lim_{x \to 0} \left[\tan x \left(\frac{1}{\sin x} \right) \right] \\
&= \lim_{x \to 0} \left[\frac{\sin x}{\cos x} \left(\frac{1}{\sin x} \right) \right] \\
&= \lim_{x \to 0} \frac{1}{\cos x} \\
&= \frac{1}{\cos 0} \\
&= 1
\end{aligned}$$

Example 6

Find $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 3x}$.

Solution

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} &= \frac{\lim\limits_{4x \to 0} 4x\left(\frac{\sin 4x}{4x}\right)}{\lim\limits_{3x \to 0} 3x\left(\frac{\sin 3x}{3x}\right)} \\
&= \frac{4}{3}\left(\frac{\lim\limits_{4x \to 0} \frac{\sin 4x}{4x}}{\lim\limits_{3x \to 0} \frac{\sin 3x}{3x}}\right) \\
&= \frac{4}{3}\left(\frac{1}{1}\right) \\
&= \frac{4}{3}
\end{aligned}
$$

Note that a calculator estimate produces the same result.

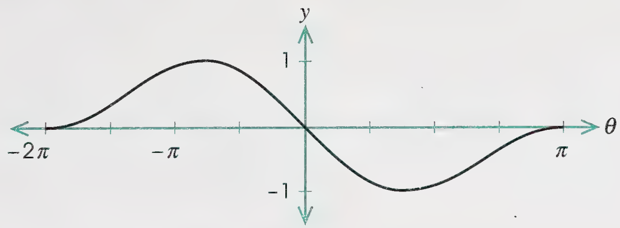
x	$\frac{\sin 4x}{\sin 3x}$
0.2	1.270 461
0.1	1.317 738
0.01	1.333 178
0.001	1.333 333

The only trigonometric limits for which you know the values involve simple sines and cosines. Therefore, to evaluate the next important limit, $\lim\limits_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$, you must write $\cos \theta$ in terms of sine. You will use the skills you learned in Section 1 to make this change.

$$
\begin{aligned}
\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta \to 0} \left[\frac{\cos \theta - 1}{\theta}\left(\frac{\cos \theta + 1}{\cos \theta + 1}\right)\right] \\
&= \lim_{\theta \to 0} \frac{\cos^2 - 1}{\theta(\cos \theta + 1)} \\
&= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} \qquad \text{(Pythagorean identity)} \\
&= \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta}\left(\frac{-\sin \theta}{\cos \theta + 1}\right)\right] \\
&= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \bullet \lim_{\theta \to 0} \frac{-\sin \theta}{\cos \theta + 1} \\
&= 1\left(\frac{-\sin 0}{\cos 0 + 1}\right) \\
&= 1\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}
$$

$\lim\limits_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$ and similarly $\lim\limits_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$

By examining the graph of $y = \frac{\cos\theta - 1}{\theta}$, notice the same limit as θ approaches 0.



You can now use these three important limits to evaluate more complex trigonometric limits.

1. Evaluate the following limits.

a. $\lim_{x\to 0} \frac{\sin 5x}{x}$ **b.** $\lim_{x\to 0} \frac{\sin \frac{1}{2}x}{x}$ **c.** $\lim_{x\to 0} \frac{\sin ax}{x}$

d. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$ **e.** $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$ **f.** $\lim_{x\to 0} x \cos x$

g. $\lim_{x\to 0} x \csc x$ **h.** $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$ **i.** $\lim_{x\to 0} \frac{\tan 2x}{3x}$

2. Evaluate the following limit by first rearranging the expression.

$$\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$$

3. Is the $\lim_{\theta\to 0} \frac{\cos\theta}{\theta}$ defined? Justify your answer with a table of values (as was done in the notes when finding $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$) or a graph.

Check your answers by turning to the Appendix.

You should now be able to apply $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and its variations.

Activity 2: Complex Trigonometric Expressions

Some of the limits to be evaluated involve more complicated expressions, requiring you to simplify or transform these expressions first.

Example 1

Evaluate $\lim_{x\to 0} x \sin \frac{1}{x}$.

Solution

Straight substitution here would result in an undefined limit. The expression must be transformed in some way so as to model $\lim_{x\to 0} \frac{\sin x}{x}$.

If you substitute $\frac{1}{x}$ with y, it would change the required limit to a similar known limit.

$$\begin{aligned}\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to 0}\frac{1}{y}\bullet\sin\ y \qquad \left(\text{Since } y=\tfrac{1}{x},\ \text{then } x=\tfrac{1}{y}.\right)\\ &= \lim_{x\to 0}\frac{\sin\ y}{y}\\ &= 1\end{aligned}$$

Example 2

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin^2\ 3x}{\sin^2\ 4x}$.

Solution

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2\ 3x}{\sin^2\ 4x} &= \lim_{x\to 0}\left(\frac{\sin\ 3x}{\sin\ 4x}\right)^2\\ &= \left(\frac{3\cancel{x}\bullet\lim\limits_{3x\to 0}\frac{\sin\ 3x}{3x}}{4\cancel{x}\bullet\lim\limits_{4x\to 0}\frac{\sin\ 4x}{4x}}\right)^2\\ &= \left[\frac{3(1)}{4(1)}\right]^2\\ &= \frac{9}{16}\end{aligned}$$

Example 3

Evaluate $\lim\limits_{x\to 0}(\sin\ x+x)$.

Solution

Since the function is not in quotient form, you may use straight substitution and evaluate.

$$\begin{aligned}\lim_{x\to 0}(\sin\ x+x) &= \lim_{x\to 0}\sin\ x+\lim_{x\to 0}x \qquad \text{(The limit of a sum is the sum of the limits.)}\\ &= \sin\ 0+0\\ &= 0+0\\ &= 0\end{aligned}$$

In some limit expressions, the angle (denoted by a variable or θ) approaches a value other than 0. Most often, all you need to do is substitute and evaluate.

Example 4

Evaluate $\lim\limits_{x\to\pi}(\sin x+\cos x)$.

Solution

$$\begin{aligned}\lim_{x\to\pi}(\sin x+\cos x)&=\lim_{x\to\pi}\sin x+\lim_{x\to\pi}\cos x\\&=\sin\pi+\cos\pi\\&=0+(-1)\\&=-1\end{aligned}$$

Example 5

Evaluate $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{\sin\theta+1}{\cos\theta+1}$.

Solution

$$\begin{aligned}\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta+1}{\cos\theta+1}&=\frac{\sin\frac{\pi}{2}+1}{\cos\frac{\pi}{2}+1}\\&=\frac{1+1}{0+1}\\&=2\end{aligned}$$

Example 6

Evaluate $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos 2x}{\cos x-\sin x}$.

Solution

This expression requires the help of a trigonometric identity since straight substitution gives $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to\frac{\pi}{4}}\frac{\cos 2x}{\cos x-\sin x}&=\lim_{x\to\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}\quad\text{(double-angle identity)}\\&=\lim_{x\to\frac{\pi}{4}}\frac{(\cos x+\sin x)\cancel{(\cos x-\sin x)}}{\cancel{\cos x-\sin x}}\\&=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\\&=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\\&=\sqrt{2}\end{aligned}$$

Example 7

Evaluate $\lim\limits_{x\to\frac{3\pi}{2}}\dfrac{\cos x}{2x}$.

Solution

$$\lim_{x \to \frac{3\pi}{2}} \frac{\cos x}{2x} = \frac{\cos \frac{3\pi}{2}}{2\left(\frac{3\pi}{2}\right)}$$
$$= \frac{0}{3\pi}$$
$$= 0$$

Read the value of $\cos \frac{3\pi}{2}$ from the unit circle or a calculator.

Example 8

Find $\lim_{x \to 0} \frac{\cos 2x}{3 \cos 3x}$.

Solution

Since the function does not yield $\frac{0}{0}$, you may use direct substitution.

$$\lim_{x \to 0} \frac{\cos 2x}{3 \cos 3x} = \frac{1}{3}\left(\lim_{x \to 0} \frac{\cos 2x}{\cos 3x}\right)$$
$$= \frac{1}{3}\left(\frac{\cos 2(0)}{\cos 3(0)}\right)$$
$$= \frac{1}{3}\left(\frac{1}{1}\right)$$
$$= \frac{1}{3}$$

Example 9

Evaluate $\lim_{x \to 0} \frac{\tan x}{\tan 2x}$.

Solution

This trigonometric function requires transformation because straight substitution gives

$$\frac{\tan 0}{\tan 2(0)} = \frac{\frac{1}{0}}{\frac{1}{0}}$$
$$= \frac{1}{0} \times \frac{0}{1}$$
$$= \frac{0}{0}$$
$$= \text{undefined}$$

Use the quotient identity for tangent to rewrite the expression.

$$\therefore \lim_{x\to 0}\frac{\tan x}{\tan 2x} = \lim_{x\to 0}\left(\frac{\frac{\sin x}{\cos x}}{\frac{\sin 2x}{\cos 2x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\bullet\frac{\cos 2x}{\sin 2x}\right)$$

$$= \lim_{x\to 0}\frac{\cos 2x}{\cos x}\bullet\lim_{x\to 0}\frac{\sin x}{\sin 2x}$$

$$= \frac{\cos 2(0)}{\cos 0}\left(\frac{\cancel{x}\lim_{x\to 0}\frac{\sin x}{x}}{2\cancel{x}\lim_{2x\to 0}\frac{\sin 2x}{2x}}\right)$$

$$= \frac{1}{1}\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)$$

$$= \frac{1}{2}$$

Note that a calculator estimate produces the same result.

x	$\frac{\tan x}{\tan 2x}$
0.5	0.350 777
0.3	0.452 156
0.2	0.479 454
0.1	0.494 966
0.01	0.499 950
0.001	0.499 999

When evaluating the limit of a trigonometric function, check straight substitution first. If the limit yields anything other than $\frac{0}{0}$, then you are finished.

If the limit yields $\frac{0}{0}$, then it is necessary to transform the function using the trigonometric identities.

1. Evaluate the following limits.

a. $\lim_{x\to 0} x\cot 3x$

b. $\lim_{x\to 0}\frac{x-\tan x}{\sin x}$

c. $\lim_{x\to\frac{\pi}{3}}\frac{\sin x}{2x}$

d. $\lim_{x\to -\frac{\pi}{2}} x^2\cos^3 x$

e. $\lim_{x\to\frac{5\pi}{3}}\sqrt{\cos x}$

f. $\lim_{x\to 2\pi}\frac{1-\sin x}{\cos x}$

g. $\lim_{x\to 0}\frac{\cos x-1}{3x}$

h. $\lim_{x\to\frac{\pi}{2}}\frac{\sin 2x}{\tan x}$

i. $\lim_{x\to\pi}\frac{\tan x-\sec x}{\sec x}$

Check your answers by turning to the Appendix.

If you encounter a limit expression in which the angle does not approach zero and where none of the previous methods work, it may be necessary to change the argument (that is, the form of the angle).

Example 10

Evaluate $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$.

Solution

Straight substitution will yield $\frac{0}{0}$ and there is no means with which to transform the function using identities. The function can be transformed to model $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ by changing $x \to \pi$ to something approaching 0.

$$x \to \pi$$
$$x - \pi \to 0$$

Therefore, $\pi - x \to 0$ is also true.

$$\sin x = \sin(\pi - x) \qquad \text{(property of sines)}$$

$$\lim_{x \to \pi} \frac{\sin x}{\pi - x} = \lim_{(x-\pi) \to 0} \frac{\sin(\pi - x)}{\pi - x}$$
$$= 1$$

This limit now models a known limit expression.

Example 11

Evaluate $\lim\limits_{\theta \to \pi} \dfrac{\cos \theta + 1}{\sin \theta}$.

Solution

Again, substitution would result in $\frac{0}{0}$. Changing the argument from $\theta \to \pi$ to something approaching 0, would give a limit function that models $\lim\limits_{x \to 0} \frac{\cos x - 1}{x} = 0$ or $\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0$.

Let $x = \pi - \theta$
$\therefore$ as $\theta \to \pi$
$(\pi - \theta) \to 0$
or $x \to 0$

$$\lim_{\theta \to \pi} \frac{\cos \theta + 1}{\sin \theta} = \lim_{x \to 0} \frac{\cos(\pi - x) + 1}{\sin(\pi - x)} \qquad (\theta = \pi - x)$$
$$= \lim_{x \to 0} \frac{-\cos x + 1}{\sin x} \qquad \text{(CAST rule)}$$
$$= \lim_{x \to 0} \left(\frac{1 - \cos x}{x} \bullet \frac{x}{\sin x} \right)$$
$$= 0(1)$$
$$= 0$$

2. Evaluate the following limits with the help of a useful substitution.

a. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$ **b.** $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\frac{\pi}{2} - x}$

Check your answers by turning to the Appendix.

You will encounter limits when developing the rules of differentiation in the next section.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Limits can be found numerically, geometrically, or algebraically. While all three of these methods will yield the same result, some limit expressions are complicated and require simplification, therefore making the algebraic method more desirable.

Some limits of trigonometric functions are easily found by simply substituting for the given value and evaluating the trigonometric expression. Often the substitution yields a quotient of the form $\frac{0}{0}$, which means that the limit of the function, in its present form, is indeterminate (may have any value). Therefore, it is necessary to change the function, with the use of identities and algebraic manipulations, to a form in which a limit can be found.

It is important to memorize the following limits:

- $\lim\limits_{x \to 0} \sin x = 0$
- $\lim\limits_{x \to 0} \cos x = 1$
- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{x}{\sin x} = 1$
- $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Note: Complicated limit expressions can be simplified using these properties.

Recall from Module 2 that there are some basic properties of limits. (Turn to Section 3, Activity 1 of Module 2 for a review of these properties.)

- The limit of a constant times a function is equal to the product of the constant and the limit of the function.
- The limit of a sum equals the sum of the limits.
- The limit of a product equals the product of the limits.
- The limit of a quotient equals the quotient of the limits.

Example 1

Evaluate $\lim\limits_{x\to 0} \frac{\cos^2 x - 1}{x^2}$.

Solution

Substitution of $x = 0$ at this point will yield $\frac{0}{0}$. Thus, the function must be transformed in order for a limit to be found.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} &= \lim_{x\to 0} \frac{-\left(1-\cos^2 x\right)}{x^2} \\
&= \lim_{x\to 0} \frac{-\sin^2 x}{x^2} \qquad \text{(Pythagorean identity)} \\
&= \lim_{x\to 0} \left[\frac{-\sin x}{x} \bullet \frac{\sin x}{x}\right] \\
&= -1(1) \\
&= -1
\end{aligned}$$

Example 2

Evaluate $\lim\limits_{x\to 0} \frac{\cos x}{1-\sin x}$.

Solution

Straight substitution will work with this function.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x}{1-\sin x} &= \frac{\cos 0}{1-\sin 0} \\
&= \frac{1}{1-0} \\
&= 1
\end{aligned}$$

Example 3

Evaluate $\lim\limits_{x\to 0} \frac{1-\cos x}{\sin x}$.

Solution

The function must be transformed in order to find a limit.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\sin x} &= \lim_{x\to 0} \left[\frac{1-\cos x}{\sin x} \bullet \frac{1+\cos x}{1+\cos x}\right] \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{\sin x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= \frac{0}{1+1} \\
&= 0
\end{aligned}$$

1. Try evaluating the following limits.

a. $\lim\limits_{x\to 0} \dfrac{\sin x}{1+\cos x}$

b. $\lim\limits_{x\to 0} \dfrac{x+\tan x}{\sin x}$

c. $\lim\limits_{x\to 0} \dfrac{\sin^2 2x}{x^2}$

d. $\lim\limits_{x\to 0} \dfrac{\sin \frac{x}{2}}{x}$

e. $\lim\limits_{x\to 0} \dfrac{\csc 2x}{\cot x}$

2. Evaluate $\lim\limits_{x\to\infty} x \sin\left(\frac{1}{x}\right)$ by replacing $\frac{1}{x}$ by θ.

Check your answers by turning to the Appendix.

Enrichment

One of the important limits that will be used in the next section is $\lim\limits_{x\to 0} \frac{\cos x-1}{x} = 0$. You have studied the proof of this limit in Activity 1 of this section. This proof required a change of cosine to sine and this was accomplished by multiplying the function by an expression equal to 1, then using the Pythagorean identity. You will now look at this limit using the double-angle identity for cosines.

Example

Evaluate the limit $\lim\limits_{x\to 0} \frac{\cos x-1}{x}$.

Solution

To change from cosine to sine, you can use the identity $\cos 2x = 1-2\sin^2 x$. Since the limit is a cosine function of x and the identity is a function of $2x$, you can replace x with $\frac{x}{2}$.

$$\cos 2\left(\frac{x}{2}\right) = 1-2\sin^2 \frac{x}{2} \text{ and } \cos x = 1-\sin^2 \frac{x}{2}$$

$$\begin{aligned}
\therefore \lim_{x\to 0} \frac{\cos x-1}{x} &= \lim_{x\to 0} \frac{\left(1-2\sin^2 \frac{x}{2}\right)-1}{x} \\
&= \lim_{x\to 0} \frac{-2\sin\frac{x}{2}\sin\frac{x}{2}}{x} \\
&= \lim_{\frac{x}{2}\to 0} \frac{-2\sin\frac{x}{2}\sin\frac{x}{2}}{2\left(\frac{x}{2}\right)} \\
&= -\frac{2}{2}\lim_{\frac{x}{2}\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \bullet \sin\frac{x}{2} \\
&= -\frac{2}{2}(1)(\sin 0) \\
&= 0
\end{aligned}$$

1. Use the double-angle identity to prove $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$.

2. Use the double-angle identity to prove $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

Check your answers by turning to the Appendix.

Conclusion

The limit of a trigonometric function has to do with how the function behaves as the angle takes on a sequence of values. A limit exists if the value of the function approaches a finite number as the angle approaches a particular value from either side.

You can sometimes determine the limit simply by evaluating the function at the specified angle. If the function is indeterminate $\left(\text{yields } \frac{0}{0}\right)$ at the specified angle, it is necessary to change the form of the function by applying identities or using algebraic manipulations to simplify.

The two fundamental trigonometric limits that you need are $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

Just as a skier becomes more proficient with practice, you can become more proficient in evaluating limits of trigonometric functions by practising the exercises in this section.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 4: Derivatives of Trigonometric Functions

Tides both fascinate and concern a lot of people. Beachcombers search through the bounty that tides bring in; the city of Venice, Italy, relies on the tides to flush out its city-wide canal system; and engineers look for economical ways to make electricity by harnessing the energy tides produce.

Tides are periodic phenomena. The height of water in a tidal region is modelled by a sine or cosine function. To generate electricity from tides, a dam traps the water of an incoming tide; the water then passes through turbines, producing electricity. It is necessary to determine the rate at which the water rises or falls. This means finding the rate of change of a trigonometric function. Rates of change are given by derivatives.

In this section you will differentiate trigonometric functions, and use those derivatives to determine rates of change. You will also examine trigonometric curves and their tangents.

Activity 1: Primary Ratios

In Module 3 you learned that a derivative of a function was the slope of the tangent to a curve and that $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x+h)-f(x)}{h}$. This is the definition of a derivative using the concept of first principles. First principles can also establish the derivatives of the trigonometric functions.

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

If $f(x) = \sin x$, then

$$\begin{aligned}
\frac{d}{dx} \sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{(sum identity for sine)} \\
&= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \bullet \frac{\cos h - 1}{h} + \cos x \bullet \frac{\sin h}{h} \right) \\
&= \lim_{h \to 0} \left(\sin x \bullet \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \bullet \frac{\sin h}{h} \right) && \text{(limit of a sum = sum of the limits)} \\
&= \sin x \bullet \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \bullet \lim_{h \to 0} \frac{\sin h}{h} && \text{(Since } \sin x \text{ and } \cos x \text{ do not involve } h\text{, they remain constant as } h \to 0.) \\
&= (\sin x)(0) + (\cos x)(1) \\
&= \cos x
\end{aligned}$$

Therefore, $\frac{d}{dx}\sin x = \cos x$.

If $y = \sin u$ is a composition of two functions, use the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\therefore \frac{d}{dx}\sin u = \cos u \bullet \frac{du}{dx}$$

Example 1

Find the derivative of $y = \sin 2x$.

Solution

You must apply the chain rule to find the derivative as well as the rule for differentiating the sine function.

$$\begin{aligned}\therefore \frac{dy}{dx} &= (\cos 2x) \bullet \frac{d}{dx}(2x) \\ &= (\cos 2x)(2) \\ &= 2\cos 2x\end{aligned}$$

Example 2

Find $\frac{dy}{dx}$ if $y = \sin\left(x^2 - 1\right)$.

Solution

$$\begin{aligned}y &= \sin\left(x^2 - 1\right) \\ \frac{dy}{dx} &= \left[\cos\left(x^2 - 1\right)\right] \bullet \frac{d}{dx}\left(x^2 - 1\right) \\ &= \cos\left(x^2 - 1\right) \bullet 2x \\ &= 2x\cos\left(x^2 - 1\right)\end{aligned}$$

Example 3

Differentiate $y = x\sin x$.

Solution

You need to use the product rule to find the derivative.

$$\begin{aligned}y &= x\sin x \\ \frac{dy}{dx} &= \frac{d}{dx}(x) \bullet \sin x + \frac{d}{dx}(\sin x) \bullet x \\ &= (1)(\sin x) + (\cos x)(x) \\ &= \sin x + x\cos x\end{aligned}$$

Example 4

Find the derivative of $y = \dfrac{x}{\sin x}$.

Solution

The quotient rule is required here.

$$\begin{aligned} y &= \frac{x}{\sin x} \\ \frac{dy}{dx} &= \frac{\sin x \bullet \frac{d}{dx}(x) - x \bullet \frac{d}{dx}(\sin x)}{\sin^2 x} \\ &= \frac{\sin x(1) - x(\cos x)}{\sin^2 x} \\ &= \frac{\sin x - x\cos x}{\sin^2 x} \end{aligned}$$

1. Find the derivative of the following sine functions, keeping in mind the need for the chain, product, and quotient rules for composite functions.

a. $y = 3\sin x$ **b.** $y = 4\sin x + x^2$

c. $y = \sin 4x$ **d.** $y = \dfrac{x^2}{\sin x}$

e. $y = \left(2 + x^2\right)\sin x$ **f.** $y = \sin^3 x$

g. $y = \sin 2x \ \sin 3x$ **h.** $y = \sin\sqrt{x}$

i. $y = \dfrac{\sin 2x}{\sin 3x}$

Check your answers by turning to the Appendix.

The derivatives of the other primary trigonometric functions can be found from first principles; but other methods prove to be shorter.

The derivative of $\cos x$ is found using its complementary angle $\left(\frac{\pi}{2} - x\right)$.

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

Recall that these cofunction identities state that the cosine of an angle is equal to the sine of its complement. Refer to Section 1, Activity 1.

$$\begin{aligned} \therefore \ \frac{d}{dx}\cos x &= \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right) \\ &= \cos\left(\frac{\pi}{2} - x\right) \bullet \frac{d}{dx}\left(\frac{\pi}{2} - x\right) \\ &= (\sin x)(-1) \\ &= -\sin x \end{aligned}$$

Therefore, $\frac{d}{dx}\cos x = -\sin x$.

Example 5

Find $\frac{dy}{dx}$ if $y = \cos^3 x$.

Solution

Remember that the function can be written as $y = (\cos x)^3$; therefore, the chain rule is needed.

$$\begin{aligned} y &= \cos^3 x \\ \frac{dy}{dx} &= 3\cos^2 x \bullet \frac{d}{dx}(\cos x) \\ &= 3\left(\cos^2 x\right)(-\sin x) \\ &= -3\cos^2 x \sin x \end{aligned}$$

Example 6

Find the derivative of $y = \cos(ax + b)$.

Solution

$$\begin{aligned} y &= \cos(ax+b) \\ \frac{dy}{dx} &= [-\sin(ax+b)] \bullet \frac{d}{dx}(ax+b) \\ &= -\sin(ax+b) \bullet a \\ &= -a \sin(ax+b) \end{aligned}$$

Example 7

Differentiate $y = \cos(\sin x)$.

Solution

Since you have a composite function, you must use the chain rule.

$$\begin{aligned} y &= \cos(\sin x) \\ \frac{dy}{dx} &= [-\sin(\sin x)] \bullet \frac{d}{dx}(\sin x) \\ &= -\sin(\sin x) \bullet \cos x \\ &= -\cos x \sin(\sin x) \end{aligned}$$

2. Use the derivatives for sine and cosine along with the rules for derivatives to differentiate the following.

a. $y = \cos 2x$

b. $y = \cos\left(2x - \frac{\pi}{2}\right)$

c. $y = \frac{\cos x}{x^2}$

d. $y = \sin x \cos x$

e. $y = \frac{\cos^2 x}{\sin^2 x}$

f. $y = \sin(\cos x)$

Check your answers by turning to the Appendix.

The derivative of $\tan x$ is $\frac{d}{dx}\tan x = \sec^2 x$, the proof of which is accomplished by using the reciprocal identity.

$$\tan x = \frac{\sin x}{\cos x}$$

Apply the quotient rule to find the derivative.

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{\cos x \bullet \frac{d}{dx}(\sin x) - \sin x \bullet \frac{d}{dx}(\cos x)}{\cos^2 x}\\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} \quad \text{(Pythagorean identity)}\\ &= \sec^2 x \quad \text{(reciprocal identity)}\end{aligned}$$

Therefore, $\frac{d}{dx}\tan x = \sec^2 x$.

In summary, the derivatives of the three primary trigonometric functions are as follows:

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$

Remember that finding the derivatives of trigonometric functions may require using the chain rule, product rule, or quotient rule in addition to the previous formulas. (You can review these by looking back at Module 3, Section 2, Activities 5 to 7.) If the trigonometric function was a composite function, these derivatives would look like the following:

- $\frac{d}{dx}\sin u = \cos u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\cos u = -\sin u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\tan u = \sec^2 u \bullet \frac{du}{dx}$

Example 8

Find the derivative of $y = x^2 \tan x$.

Solution

$$\begin{aligned}y &= x^2 \tan x\\ \frac{dy}{dx} &= x^2 \bullet \frac{d}{dx}(\tan x) + \tan x \bullet \frac{d}{dx}\left(x^2\right) \quad \text{(product rule)}\\ &= x^2 \sec^2 x + 2x \tan x\end{aligned}$$

Example 9

Find $\frac{dy}{dx}$ if $y = \sin x + \tan x$.

Solution

$$y = \sin x + \tan x$$
$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\tan x)$$
$$= \cos x + \sec^2 x$$

Example 10

Differentiate $y = \dfrac{\tan^2 x}{x}$.

Solution

$$y = \frac{\tan^2 x}{x}$$
$$\frac{dy}{dx} = \frac{x \bullet 2 \tan x \bullet \frac{d}{dx}(\tan x) - \tan^2 x \bullet \frac{d}{dx}(x)}{x^2}$$ (chain rule and quotient rule)
$$= \frac{2x \tan x \sec^2 x - \tan^2 x}{x^2}$$

Example 11

Differentiate $y = \sqrt{\tan 2x}$.

Solution

$$y = \sqrt{\tan 2x}$$
$$\frac{dy}{dx} = \frac{1}{2}(\tan 2x)^{-\frac{1}{2}} \bullet \frac{d}{dx}(\tan 2x)$$
$$= \frac{\sec^2 2x \bullet 2}{2\sqrt{\tan 2x}}$$
$$= \frac{\sec^2 2x}{\sqrt{\tan 2x}}$$

3. Use the derivatives of the three primary trigonometric ratios and the rule for derivatives of composite functions to differentiate the following:

   a. $y = \dfrac{1}{\tan x + 1}$

   b. $y = \tan^2 x$

   c. $y = \tan x^2$

   d. $y = \cos(\tan x)$

   e. $y = \tan \sqrt{x}$

Check your answers by turning to the Appendix.

Activity 2: Reciprocal Ratios

In the previous activity you found the derivatives of the primary trigonometric ratios. In this activity you will determine the derivatives of their reciprocal ratios using your knowledge of primary ratios.

You know that $\csc x = \frac{1}{\sin x}$. This fact, along with the quotient rule, will generate the derivative of $\csc x$.

$$\csc x = \frac{1}{\sin x}$$

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{\sin x \bullet \frac{d}{dx}(1) - 1 \bullet \frac{d}{dx}(\sin x)}{\sin^2 x}\\ &= \frac{\sin x \bullet 0 - 1 \bullet \cos x}{\sin^2 x}\\ &= \frac{-\cos x}{\sin^2 x}\\ &= \frac{-1}{\sin x}\left(\frac{\cos x}{\sin x}\right) \quad \text{(Regroup to simplify.)}\\ &= -\csc x \cot x\end{aligned}$$

Therefore, $\frac{d}{dx}\csc x = -\csc x \cot x$.

The derivatives of the other two reciprocal ratios are as follows:

- $\dfrac{d}{dx}\sec x = \sec x \tan x$
- $\dfrac{d}{dx}\cot x = -\csc^2 x$

The proofs of these derivatives are left for you to practise.

Example 1

Differentiate $y = \sec\left(x^2 + 1\right)$.

Solution

$$y = \sec\left(x^2 + 1\right)$$

$$\begin{aligned}\frac{dy}{dx} &= \sec\left(x^2 + 1\right) \tan\left(x^2 + 1\right) \bullet 2x\\ &= 2x \sec\left(x^2 + 1\right) \tan\left(x^2 + 1\right)\end{aligned}$$

Example 2

Find $\frac{dy}{dx}$ if $y = \cot\sqrt{x}$.

Solution

$$y = \cot\sqrt{x}$$

$$\begin{aligned}\frac{dy}{dx} &= -\csc^2\sqrt{x} \bullet \frac{d}{dx}\left(\sqrt{x}\right)\\ &= -\csc^2\sqrt{x} \bullet \frac{1}{2}x^{-\frac{1}{2}}\\ &= \frac{-\csc^2\sqrt{x}}{2\sqrt{x}}\end{aligned}$$

Example 3

Find the derivative of $y = 2\csc 3x$.

Solution

$$y = 2\csc 3x$$

$$\frac{dy}{dx} = 2 \bullet -\csc 3x \cot 3x \bullet \frac{d}{dx}(3x)$$

$$= -6\csc 3x \cot 3x$$

Example 4

Find $\frac{dy}{dx}$ if $y = 2\sec^2\left(2x^3\right)$.

Solution

$$y = 2\sec^2\left(2x^3\right)$$

$$\frac{dy}{dx} = 4\sec\left(2x^3\right) \bullet \frac{d}{dx}\left[\sec\left(2x^3\right)\right]$$

$$= 4\sec\left(2x^3\right) \bullet \sec\left(2x^3\right) \tan\left(2x^3\right) \bullet \frac{d}{dx}\left(2x^3\right)$$

$$= 4\sec\left(2x^3\right) \bullet \sec\left(2x^3\right) \tan\left(2x^3\right) \bullet 6x^2$$

$$= 24x^2 \sec^2\left(2x^3\right) \tan\left(2x^3\right)$$

1. Use the derivatives of the primary ratios and the quotient rule to prove the following.

a. $\frac{d}{dx}\sec x = \sec x \tan x$

b. $\frac{d}{dx}\cot x = -\csc^2 x$

2. Differentiate each function.

a. $y = \csc^2 2x$

b. $y = \sec x \tan x$

c. $y = \sec^2 x - \tan^2 x$

d. $y = x^2 \cot x$

e. $y = \frac{\csc 2x}{2x}$

Check your answers by turning to the Appendix.

In summary, the derivatives of the six trigonometric functions are as follows:

- $\frac{d}{dx}\sin u = \cos u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\cos u = -\sin u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\tan u = \sec^2 u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\csc u = -\csc u \cot u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\sec u = \sec u \tan u \bullet \frac{du}{dx}$
- $\frac{d}{dx}\cot u = -\csc^2 u \bullet \frac{du}{dx}$

These derivatives should be memorized. Note that the “co” functions (cosine, cosecant, and cotangent) have derivatives that are negative.

In order to find these derivatives, you had to first evaluate the two limits.

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \text{ and } \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

3. Use all of this information to find $\frac{dy}{dx}$ for each of the following:

a. $y=\sin^2 x+\tan^2 x$

b. $y=\dfrac{\sin x}{\sec x}$

c. $y=\csc^2 x-\cot^2 x$

d. $y=(\csc x+\cot x)^2$

e. $y=3\csc x\cos^3 x$

Check your answers by turning to the Appendix.

Remember: The derivative of a cofunction (cosine, cosecant, or cotangent) is negative.

Activity 3: Complex Trigonometric Expressions

Trigonometric expressions may contain a multiple of functions and operations. You must take great care to ensure accuracy and thoroughness when finding their derivatives.

You have had some practise in finding the derivatives of trigonometric expressions where it was necessary to use the chain, product, and quotient rules. You will continue to use these rules and other techniques to differentiate and then simplify trigonometric expressions.

Example 1

Find the derivative of $y=\sqrt{\csc 2x}$.

Solution

This function is composed of three functions: square root, cosecant, and multiplying by 2. The chain rule must be used. It is easiest to apply it one step at a time.

$$y=\sqrt{\csc 2x}$$

$$y=(\csc 2x)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}(\csc 2x)^{-\frac{1}{2}}\bullet\frac{d}{dx}(\csc 2x)$$

$$=\frac{1}{2}(\csc 2x)^{-\frac{1}{2}}(-\csc 2x\cot 2x)\frac{d}{dx}(2x)$$

$$=\frac{-2\csc 2x\cot 2x}{2\sqrt{\csc 2x}}\bullet\frac{\sqrt{\csc 2x}}{\sqrt{\csc 2x}} \quad \text{(Rationalize the denominator to simplify.)}$$

$$=\frac{-\cancel{\csc 2x}\cot 2x\sqrt{\csc 2x}}{\cancel{\csc 2x}}$$

$$=-\cot 2x\sqrt{\csc 2x}$$

For some complex trigonometric expressions, it is best to simplify first and then differentiate.

Example 2

Find the derivative and simplify $y=\dfrac{\sin^2 x}{(1-\cos x)^2}$.

Solution

$$y=\frac{\sin^2 x}{(1-\cos x)^2}$$

You may want to simplify this function before differentiating. The Pythagorean identity allows you to rewrite $\sin^2 x$ as $1-\cos^2 x$.

$$y=\frac{1-\cos^2 x}{(1-\cos x)^2}$$

Factor the numerator to cancel a common factor.

$$\begin{aligned} y&=\frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}(1-\cos x)}\\ &=\frac{1+\cos x}{1-\cos x}\end{aligned}$$

Now, take the derivative of this simpler function using the quotient rule.

$$\begin{aligned}\frac{dy}{dx}&=\frac{(1-\cos x)\frac{d}{dx}(1+\cos x)-(1+\cos x)\frac{d}{dx}(1-\cos x)}{(1-\cos x)^2}\\ &=\frac{(1-\cos x)(0-\sin x)-(1+\cos x)(0+\sin x)}{(1-\cos x)^2}\\ &=\frac{(-\sin x+\sin x\cos x)-(\sin x+\sin x\cos x)}{(1-\cos x)^2}\\ &=\frac{-2\sin x}{(1-\cos x)^2}\end{aligned}$$

Example 3

If $y=\sin\left(2x^2+4x+6\right)$, find $\frac{dy}{dx}$.

Solution

Remember that $\frac{d}{dx}\sin u=\cos u\bullet\frac{du}{dx}$.

Let $u=2x^2+4x+6$.

$$\begin{aligned}\frac{dy}{dx}&=\cos\left(2x^2+4x+6\right)\bullet\frac{d}{dx}\left(2x^2+4x+6\right)\\ &=\cos\left(2x^2+4x+6\right)\bullet(4x+4)\\ &=(4x+4)\cos\left(2x^2+4x+6\right)\end{aligned}$$

Example 4

Use the chain rule to find $\frac{dy}{dx}$ if $y = 6u^4$ and $u = \sin x + \cos x$.

Solution

You must first make the appropriate substitutions.

$$\begin{aligned} y &= 6(\sin x + \cos x)^4 \\ \frac{dy}{dx} &= 24(\sin x + \cos x)^3 \bullet \frac{d}{dx}(\sin x + \cos x) \\ &= 24(\sin x + \cos x)^3 (\cos x - \sin x) \\ &= 24(\sin x + \cos x)^2 (\sin x + \cos x)(\cos x - \sin x) \\ &= 24\left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right)\left(\cos^2 x - \sin^2 x\right) \\ &= 24\left(\sin^2 x + \cos^2 x + 2\sin x \cos x\right)\left(\cos^2 x - \sin^2 x\right) \\ &= 24(1 + \sin 2x)\cos 2x \quad \text{(Pythagorean and double-angle identities)} \end{aligned}$$

Example 5

Find the derivative of $y = \dfrac{\sin(2x+1)}{x^2+1} + 2\tan x$.

Solution

Remember that the derivative of a sum is the sum of the derivatives.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\sin(2x+1)}{x^2+1}\right)+\frac{d}{dx}(2\tan x)$$

The quotient rule must be used to differentiate the first part.

$$=\frac{\left(x^2+1\right)\frac{d}{dx}\left[\sin(2x+1)\right]-\sin(2x+1)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}+2\frac{d}{dx}(\tan x)$$

$$=\frac{2\left(x^2+1\right)\cos(2x+1)-2x\sin(2x+1)}{\left(x^2+1\right)^2}+2\sec^2 x$$

Don't forget to use the chain rule.

As complicated as this looks, it is as simplified as it can be.

1. Find the derivative of each of the following, and simplify as much as possible.

a. $y=\dfrac{\sin x+2\cos x}{\sin x-2\cos x}$

b. $y=9x+3\cot 3x-\cot^3 3x$

c. $y=\dfrac{u^5}{5}-\dfrac{u^3}{5}$, when $u=\tan x$

d. $y=\dfrac{\cos 3x}{x^2+2}+\tan 2x$

e. $y=\sqrt{\sin 2x+x}+\dfrac{\csc 3x}{x^3+1}$

Check your answers by turning to the Appendix.

An equation that defines y as a function of x is said to be in explicit form when it is in the form of "y equals", with a single y on one side of the equal sign and none on the other side. Any other form of the equation is an implicit form.

Type of Equation	Implicit Form	Explicit Form
Algebraic	$xy+1=2x-y$	$y=\frac{2x-1}{x+1}$
Trigonometric	$\tan y=\sin x$	$y=\tan x \sin x$

In many cases, particularly with trigonometric functions, the equation cannot be rewritten in explicit form. You must use a method called implicit differentation (see Section 2, Activity 8 of Module 3) to find $\frac{dy}{dx}$ when y is defined implicitly.

Recall that if $y^2+xy=x$, differentiating y with respect to x implicitly looks as follows:

$$2y \bullet \frac{dy}{dx}+\left(x \bullet \frac{dy}{dx}+y \bullet 1\right)=1$$

Use the product rule for the xy term.

$$\frac{dy}{dx}(2y+x)+y=1$$

$$\frac{dy}{dx}=\frac{1-y}{2y+x}$$

Example 6

Find the derivative of $x=\sin y$ using implicit differentiation.

Solution

You must find the derivative, with respect to x, of each term in the expression.

$$\frac{d}{dx}(x)=\frac{d}{dx}(\sin y)$$

$$1=\cos y \bullet \frac{dy}{dx}$$

Now isolate the $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

Therefore, the derivative of $x=\sin y$ with respect to x is $\frac{1}{\cos y}$.

Example 7

Find $\frac{dy}{dx}$ implicitly if $\cos xy = x$.

Solution

$$\cos xy = x$$

$$\frac{d}{dx}(\cos xy) = \frac{d}{dx}(x)$$

Use the chain rule on the left side.

$$(-\sin xy) \bullet \frac{d}{dx}(xy) = 1$$

$$-\sin xy\left(x \bullet \frac{dy}{dx} + y \bullet 1\right) = 1$$

Carefully isolate $\frac{dy}{dx}$.

$$x \bullet \frac{dy}{dx} + y = \frac{1}{-\sin xy}$$

$$x \bullet \frac{dy}{dx} = -\frac{1}{\sin xy} - y$$

$$\frac{dy}{dx} = -\frac{1}{x \sin xy} - \frac{y}{x}$$

Example 8

Differentiate $\tan y = \sin x$ implicitly.

Solution

$$\tan y = \sin x$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(\sin x)$$

$$\sec^2 y \bullet \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{\sec^2 y}$$

$$= \cos x \cos^2 y \qquad \text{(reciprocal identity)}$$

Example 9

Find $\frac{dy}{dx}$ if $y^2 - x^2 = \cos x$.

Solution

$$y^2 - x^2 = \cos x$$

$$\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}\left(x^2\right) = \frac{d}{dx}(\cos x)$$

$$2y \bullet \frac{dy}{dx} - 2x = -\sin x$$

$$2y \bullet \frac{dy}{dx} = -\sin x + 2x$$

$$\frac{dy}{dx} = \frac{2x - \sin x}{2y}$$

Example 10

Find the derivative of $\sin x + \cos y = xy$ implicitly.

Solution

$$\sin x + \cos y = xy$$

$$\frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos y) = \frac{d}{dx}(xy)$$

$$\cos x + (-\sin y) \bullet \frac{dy}{dx} = x \bullet \frac{dy}{dx} + y \bullet 1 \qquad \text{(Use the product rule.)}$$

$$-\sin y \bullet \frac{dy}{dx} - x \bullet \frac{dy}{dx} = y - \cos x$$

$$\frac{dy}{dx}(-\sin y - x) = y - \cos x$$

$$\frac{dy}{dx} = \frac{y - \cos x}{-\sin y - x}$$

Example 11

Find the derivative of $\sec(x+y) = \tan x - \tan y$.

Solution

$$
\begin{aligned}
\sec(x+y) &= \tan x - \tan y \\
\frac{d}{dx}\left[\sec(x+y)\right] &= \frac{d}{dx}(\tan x) - \frac{d}{dx}(\tan y) \\
\sec(x+y)\tan(x+y) \bullet \frac{d}{dx}(x+y) &= \sec^2 x - \sec^2 y \bullet \frac{dy}{dx} \qquad \text{(Use the chain rule.)} \\
\sec(x+y)\tan(x+y) \bullet \left(1 + \frac{dy}{dx}\right) &= \sec^2 x - \sec^2 y \bullet \frac{dy}{dx} \\
\sec(x+y)\tan(x+y) + \sec(x+y)\tan(x+y) \bullet \frac{dy}{dx} &= \sec^2 x - \sec^2 y \bullet \frac{dy}{dx} \\
\sec(x+y)\tan(x+y) \bullet \frac{dy}{dx} + \sec^2 y \bullet \frac{dy}{dx} &= \sec^2 x - \sec(x+y)\tan(x+y) \\
\frac{dy}{dx}\left[\sec(x+y)\tan(x+y) + \sec^2 y\right] &= \sec^2 x - \sec(x+y)\tan(x+y) \\
\frac{dy}{dx} &= \frac{\sec^2 x - \sec(x+y)\tan(x+y)}{\sec(x+y)\tan(x+y) + \sec^2 y}
\end{aligned}
$$

2. Use implicit differentiation to find the derivatives of the following, and simplify the final expressions as much as possible.

a. $\sin\ xy = x$

b. $\tan\ (x+y) = y$

c. $y^2 - xy = \sin\ x$

d. $2x^2y + y^2 = \cos\ x$

e. $x + y = \cot\ (x-y)$

f. $x^2 - x\cos\ y + y^3 = 0$

Check your answers by turning to the Appendix.

As you have probably noticed, the final form of the answer is somewhat arbitrary and depends on the identities and how they are applied.

Activity 4: Trigonometric Curves and Their Tangents

One of the uses of the derivative, as you studied in Module 3, is its connection to the slope of the tangent line. In this activity you will find the slopes and equations of tangent lines, at given points on a trigonometric curve, using the definition of a derivative.

You will begin this activity by looking at a graphical representation of the derivatives of the sine and cosine functions.

Since the derivative of $\sin\ x$ is $\cos\ x$ if x is measured in radians, the slope of the sine function for any value of x equals the value of $\cos\ x$.

The graphical representation of $\frac{d}{dx}\sin\ x = \cos\ x$ is as follows:

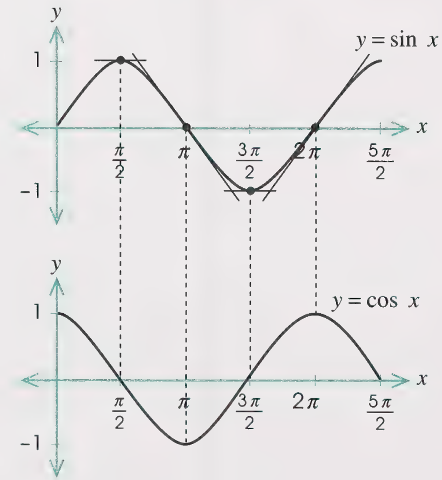
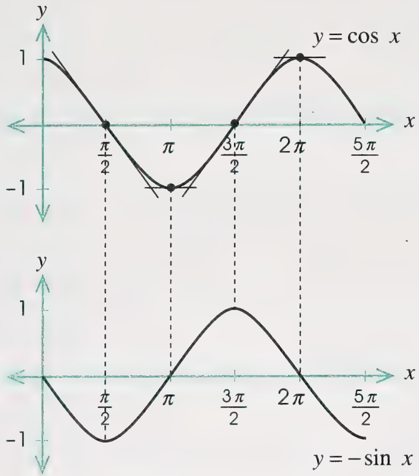


Fig 26

In the previous diagram, the slope of $y = \sin\ x$ at any value of x equals the value of the function $y = \cos\ x$ for that same value of x. In the graph of $y = \sin\ x$ there are four tangent lines drawn. By observation, note the slope of $\sin\ x$ is zero when $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. In the graph of $y = \cos\ x$ the value of $\cos\ x$ is zero when $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. The slope of $\sin\ x$ is $+1$ when $x = 0$ and 2π, and the slope is -1 when $x = \pi$. This correspondence between the slope of the graph of $y = \sin\ x$ and the value of $\cos\ x$ occurs for every value of x (if x is measured in radians).

Now look at the graphical representation of $\frac{d}{dx}\cos\ x = -\sin\ x$.



There are tangent lines drawn in four places on the graph of $y = \cos\ x$. The slopes of these tangents equal the values of $-\sin\ x$, as shown in the previous diagram. The slope of a tangent to $\cos\ x$ is zero when $x = \pi$ and 2π. In the diagram $-\sin\ x$ is zero when $x = \pi$ and 2π. Also note that the slope of $y = \cos\ x$ is $+1$ at $x = \frac{3\pi}{2}$, and is -1 at $x = \frac{\pi}{2}$. This correspondence between the slope of the graph of $y = \cos\ x$ and the value of $-\sin\ x$ occurs for every value of x.

These two relationships model the derivatives of sine and cosine.

$$\frac{d}{dx}\sin\ x = \cos\ x \text{ and } \frac{d}{dx}\cos\ x = -\sin\ x$$

Once you have determined the slope of the tangent at a particular point on the curve, you can find the equation of the tangent line using the point-slope form of the line using that point. That is, given a slope m and a point (x_1, y_1), the equation of the line is

$y - y_1 = m(x - x_1)$.

You have been shown graphically that the slope of the tangent to $y = \sin\ x$ at $x = 0$ is 1. Now, the first example will work through it algebraically.

Example 1

For the graph of the curve $y = \sin\ x$, find the slope of the tangent where $x = 0$; then determine the equation of that tangent line.

Solution

The slope of the tangent is given by the value of the derivative at $x = 0$.

$$y = \sin\ x$$
$$\frac{dy}{dx} = \cos\ x$$

When $x = 0$, $\frac{dy}{dx} = \cos\ 0$

$$= 1$$

Therefore, the slope $m=1$.

For the function $y=\sin\ x$, when $x=0$, $y=\sin\ 0=0$.

Therefore, the tangent to $y=\sin\ x$ at $x=0$ has a slope of 1, and passes through the point $(0,0)$.

$$\begin{aligned}\therefore\ y-y_1 &= m(x-x_1)\\ y-0 &= 1(x-0)\\ y &= x\end{aligned}$$

The equation of the required tangent is $y=x$.

Example 2

Find the slope and the equation of the tangent to $y=x+\cos\ x$ at $x=\frac{3\pi}{2}$.

Solution

$$y=x+\cos\ x$$

$$\frac{dy}{dx}=1-\sin\ x$$

The slope of this function, where $x=\frac{3\pi}{2}$, is the value of the derivative at $x=\frac{3\pi}{2}$.

$$\begin{aligned}\frac{dy}{dx}&=1-\sin\ \frac{3\pi}{2}\\ &=1-(-1)\\ &=2\end{aligned}$$

The slope of the tangent is 2.

When $x=\frac{3\pi}{2}$, $y=\frac{3\pi}{2}+\cos\ \frac{3\pi}{2}$ (Read $\cos\ \frac{3\pi}{2}$ from the unit circle.)

$$\begin{aligned}&=\frac{3\pi}{2}+0\\ &=\frac{3\pi}{2}\end{aligned}$$

The tangent has a slope of 2, and passes through the point $\left(\frac{3\pi}{2},\frac{3\pi}{2}\right)$.

$$\begin{aligned}\therefore\ y-y_1 &= m(x-x_1)\\ y-\frac{3\pi}{2} &= 2\left(x-\frac{3\pi}{2}\right)\\ y-\frac{3\pi}{2} &= 2x-3\pi\\ y &= 2x-3\pi+\frac{3\pi}{2}\\ &= 2x-\frac{3\pi}{2}\ \text{ or }\ 4x-2y-3\pi=0\end{aligned}$$

The required tangent has a slope of 2 and is defined by $4x-2y-3\pi=0$.

Example 3

Approximate to two decimal places the slope of the tangent line which can be drawn to the graph of $y = 2\sin x - \cos x$ at $x = -1$. Determine the equation of the tangent.

Solution

Note that the value of $x = -1$ means 1 rad in the negative direction.

$$y = 2\sin x - \cos x$$
$$\frac{dy}{dx} = 2\cos x + \sin x$$

At $x = -1$, $\frac{dy}{dx} = 2\cos(-1) + \sin(-1)$
$$\doteq 2(0.540\,302\,305) + (-0.841\,470\,984)$$
$$\doteq 0.24$$

The slope of the tangent is approximately 0.24.

At $x = -1$, $y = 2\sin(-1) - \cos(-1)$
$$\doteq -2.223\,244\,276$$

The tangent to the curve at $x = -1$ has a slope of approximately 0.24, and passes through $(-1, -2.22)$.

$$\therefore\ y - y_1 = m(x - x_1)$$
$$y + 2.22 \doteq 0.24(x+1)$$
$$y \doteq 0.24x + 0.24 - 2.22$$
$$y \doteq 0.24x - 1.98 \quad \text{or} \quad 0.24x - y - 1.98 \doteq 0$$

The required tangent has a slope of 0.24 and is defined by $0.24x - y - 1.98 \doteq 0$.

Example 4

Find an equation of the line tangent to $y = 3(1 - \sin 2x)$ at $x = \pi$.

Solution

At $x = \pi$, $y = 3(1 - \sin 2\pi)$
$$= 3(1-0)$$
$$= 3$$

Therefore, the point of tangency is $(\pi, 3)$.

Differentiate to find the slope.

$$
\begin{aligned}
y &= 3(1-\sin 2x) \\
\frac{dy}{dx} &= 3 \bullet \frac{d}{dx}(1-\sin 2x) \\
&= 3(0-\cos 2x) \bullet \frac{d}{dx}(2x) \\
&= -6\cos 2x
\end{aligned}
$$

$$
\begin{aligned}
\text{At } x=\pi,\ \frac{dy}{dx} &= -6\cos 2\pi \\
&= -6(1) \\
&= -6
\end{aligned}
$$

Find the equation of the tangent line with a slope of -6, passing through the point $(\pi, 3)$.

$$
\begin{aligned}
y-y_1 &= m(x-x_1) \\
y-3 &= -6(x-\pi) \\
y &= -6x+6\pi+3 \quad \text{or} \quad 6x+y-6\pi-3=0
\end{aligned}
$$

Example 5

Find the points at which the graph of $y=\sin x$ has a horizontal tangent.

Solution

Remember that if a line is horizontal, then its slope is zero. Therefore, the derivative of the function equals zero.

$$
\begin{aligned}
y &= \sin x \\
\frac{dy}{dx} &= \cos x \\
&= 0
\end{aligned}
$$

You must find the angles for which cosine is 0. This occurs at $\frac{\pi}{2}$ rad, $\frac{3\pi}{2}$ rad, $\frac{5\pi}{2}$ rad, and so on.

Therefore, all the points where $y=\sin x$ has a horizontal tangent are $\frac{\pi}{2}+n\pi$, where n is an integer.

1. Use the rules for derivatives to find the slopes and equations of tangent lines for the following conditions.

a. $y=\sin x$ at $x=\frac{\pi}{3}$

b. $y=\tan^3 x$ at $x=\frac{\pi}{4}$

c. $y=\sec x$ at $\left(\frac{\pi}{3}, 2\right)$

d. $y=\cos 2x$ at $x=\frac{2\pi}{3}$

e. $y=x-\sin x$ at $x=\frac{\pi}{2}$

Use a calculator to answer question 2.

2. Determine the slope and the equation of the tangent of $y = \sin\ x$ at $x = 16.3$. Round your answer to two decimal places.

Check your answers by turning to the Appendix.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Finding the derivatives of the six trigonometric functions can become more complicated when they are combined with each other and with other algebraic functions. The derivatives of these composite functions require the use of the rules for differentiation as well as the derivatives of trigonometric functions. The following list is a summary of the differentiation rules you studied in Module 3 and in this section. To become skilled at differentiation, it is recommended that you memorize each rule.

General Rules (If *u* and *v* are differentiable functions of *x*.)

- constant multiple rule: $\frac{d}{dx}(cu) = c \bullet \frac{d}{dx}(u)$
- sum or difference rule: $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$
- product rule: $\frac{d}{dx}(uv) = u \bullet \frac{d}{dx}(v) + v \bullet \frac{d}{dx}(u)$
- quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \bullet \frac{d}{dx}(u) - u \bullet \frac{d}{dx}(v)}{v^2}$
- chain rule: $\frac{d}{dx}(u)^n = nu^{n-1} \bullet \frac{d}{dx}(u)$

Derivatives of Algebraic Functions

- constant rule: $\frac{d}{dx}(c) = 0$
- power rule: $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Derivatives of Trigonometric Functions

- $\frac{d}{dx}(\sin\ x) = \cos\ x$
- $\frac{d}{dx}(\tan\ x) = \sec^2\ x$
- $\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x$
- $\frac{d}{dx}(\cos\ x) = -\sin\ x$
- $\frac{d}{dx}(\csc\ x) = -\csc\ x\ \cot\ x$
- $\frac{d}{dx}(\cot\ x) = -\csc^2\ x$

Finding the derivatives of composite functions requires you to use a combination of these rules.

Example 1

Find $\frac{dy}{dx}$ if $y = x - \tan x$.

Solution

The function is a difference between an algebraic term and a trigonometric function; thus, you must use the difference rule.

$$\begin{aligned} y &= x - \tan x \\ \frac{dy}{dx} &= \frac{d}{dx}(x) - \frac{d}{dx}(\tan x) \\ &= 1 - \sec^2 x \end{aligned}$$

Example 2

Find the derivative of $y = x \sec x$.

Solution

This function contains two terms that are multiplied; therefore, it requires the product rule.

$$\begin{aligned} y &= x \sec x \\ \frac{dy}{dx} &= x \bullet \frac{d}{dx}(\sec x) + \sec x \bullet \frac{d}{dx}(x) \\ &= x(\sec x \tan x) + \sec x(1) \\ &= \sec x(x \tan x + 1) \end{aligned}$$

Example 3

If $y = \cot^3 2x$, find $\frac{dy}{dx}$.

Solution

The function can be rewritten as $y = (\cot 2x)^3$. You must use the chain rule, finding the derivative of the "outside" and "inside".

$$\begin{aligned} \frac{dy}{dx} &= 3\cot^2 2x \bullet \frac{d}{dx}(\cot 2x) \\ &= \left(3\cot^2 2x\right)\left(-\csc^2 2x\right)\frac{d}{dx}(2x) \\ &= -6\cot^2 2x \csc^2 2x \end{aligned}$$

Example 4

Use the quotient rule to find the derivative of $y = \frac{1+\csc x}{1-\csc x}$.

Solution

$$y = \frac{1+\csc x}{1-\csc x}$$

$$\frac{dy}{dx} = \frac{(1-\csc x)\bullet \frac{d}{dx}(1+\csc x) - (1+\csc x)\bullet \frac{d}{dx}(1-\csc x)}{(1-\csc x)^2}$$

$$= \frac{(1-\csc x)\bullet(-\csc x \cot x) - (1+\csc x)\bullet(\csc x \cot x)}{(1-\csc x)^2}$$

$$= \frac{-\csc x \cot x + \csc^2 x \cot x - \csc x \cot x - \csc^2 x \cot x}{(1-\csc x)^2}$$

$$= \frac{-2\csc x \cot x}{(1-\csc x)^2}$$

Because of the many trigonometric identities, the derivative of a trigonometric function can take many forms. This can present a challenge when you are trying to match your answer with the one given in the Appendix. You must sometimes simplify your derivative after differentiating. The following examples illustrate this.

Example 5

Differentiate and simplify $y = \frac{2}{3}\sin^{\frac{3}{2}} x - \frac{2}{7}\sin^{\frac{7}{2}} x$.

Solution

$$\frac{dy}{dx} = \sin^{\frac{1}{2}} x \cos x - \sin^{\frac{5}{2}} x \cos x$$

$$= \left(\sin^{\frac{1}{2}} x - \sin^{\frac{5}{2}} x\right)\cos x$$

The derivative has been found, now you must simplify.

$$= \sin^{\frac{1}{2}} x\left(1-\sin^{\frac{4}{2}} x\right)\cos x \qquad \text{(Factor.)}$$

$$= \sqrt{\sin x}\left(1-\sin^2 x\right)\cos x$$

$$= \sqrt{\sin x}\left(\cos^2 x\right)\cos x \qquad \text{(Pythagorean identity)}$$

$$= \cos^3 x\sqrt{\sin x}$$

Example 6

Find $\frac{dy}{dx}$, in its simplest form, if $y = 1 - \cos 2x + 2\cos^2 x$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(1) - \frac{d}{dx}(\cos 2x) + 2 \bullet \frac{d}{dx}(\cos x)^2$$

$$= \underbrace{0 + \sin 2x \bullet \frac{d}{dx}(2x)}_{\text{chain rule}} + \underbrace{2 \bullet 2\cos x \bullet \frac{d}{dx}(\cos x)}_{\text{constant, power, and chain rules}}$$

$$= 2\sin 2x + 4\cos x(-\sin x)$$

$$= 4\sin x \cos x - 4\cos x \sin x \qquad \text{(double-angle identity)}$$

$$= 0$$

1. Determine the derivatives for the following:

 a. $y = 2x\cos x - 2\sin x$ **b.** $y = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}$

2. For each function, find the derivative. Simplify your answers.

 a. $y = \csc 3x + \cot 3x$ **b.** $y = \sin x(\sin x + \cos x)$

Check your answers by turning to the Appendix.

Enrichment

In Section 2: Activity 9 of Module 3, you studied higher-order derivatives. As with algebraic functions, higher-order derivatives of trigonometric functions can be found. You will apply these in the next module. The following table summarizes the notation for higher-order derivatives.

Function	First Derivative	Second Derivative	Third Derivative
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$

Example

If $y = 2\sin x$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

Solution

$$\frac{dy}{dx} = 2 \bullet \frac{d}{dx}(\sin x)$$
$$= 2\cos x$$

$$\frac{d^2y}{dx^2} = 2 \bullet \frac{d}{dx}(\cos x)$$
$$= 2(-\sin x)$$
$$= -2\sin x$$

$$\frac{d^3y}{dx^2} = -2 \bullet \frac{d}{dx}(\sin x)$$
$$= -2\cos x$$

Try the following.

1. If $y = -2 \cos x$, find $\frac{d^2 y}{dx^2}$.

2. For $y = \sin x$, determine the first four derivatives. What are the next four higher derivatives?

3. Repeat question 2 for the function $y = \cos x$.

4. Evaluate $\frac{d^4 y}{dx^4}$ at $x = \frac{\pi}{2}$ if $y = 2 \sin x$.

5. Find the second derivative of $\sin y = \cos x$ implicitly.

Check your answers by turning to the Appendix.

Conclusion

In this last section of the module you were able to put to use the concepts studied in the third section. The derivative of any function is defined as its limit as the variable approaches zero. The derivative of the sine function was determined using first principles. From the derivative of the sine function and using the rules for derivatives, you were then able to derive the derivatives of the other five trigonometric functions. You then applied the derivatives of the three primary and three reciprocal trigonometric functions to composite functions. These more complex functions combined algebraic and trigonometric functions. You also applied the rules for implicit differentiation to trigonometric functions. In the last section, an application of the derivative—slopes and equations of tangent lines—was studied.

You will encounter further applications of these derivatives in the coming modules and in future, higher-level studies of calculus.

In particular, you will model periodic motion such as the tides in Venice, Italy using trigonometric function. The analysis of that motion will involve an application of the derivative techniques that you studied in this section.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

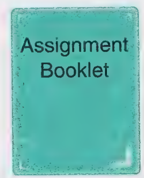
Module Summary

In this module you have reviewed and extended your knowledge of trigonometric identities and equations. Facility with these concepts enabled you to move on to study the calculus of trigonometry—the limits and derivatives of trigonometric functions.

The graphs of the sine and cosine functions appear like undulating waves on a lake. Trigonometry may be used in the sciences to model wave motion and other periodic events. The behaviour of alternating current, tidal action, the rotation of a Ferris wheel, the vibration of a violin string are but a few of the real-world situations to which the calculus of trigonometry is applied.

You will encounter futher applications in coming modules. You have just scratched the surface of this branch of calculus. Having gained an understanding of the concepts to this point provides a good base for future study of this topic.

Final Module Assignment



You are now ready to complete the final module assignment.

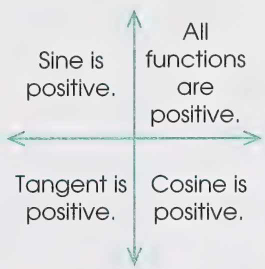
APPENDIX

Glossary

Suggested Answers

Glossary

CAST rule: a rule which determines the signs of the primary trigonometric functions in relationship to their positions on the coordinate plane



Cofunction identities: the cosine of an angle is equal to the sine of its complementary angle, and the sine of an angle is equal to the cosine of its complement

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\,\theta \qquad \sin\left(\frac{\pi}{2}-\theta\right)=\cos\,\theta$$

Distributive property: the product of a term and a sum can be written as the sum of two products

For example, $5(3+4)=5(3)+5(4)$.

Pythagorean identities: identities that are established by applying the Pythagorean Theorem to a right triangle on the unit circle

$$\sin^2\theta+\cos^2\theta=1 \qquad \tan^2\theta+1=\sec^2\theta$$
$$\cot^2\theta+1=\csc^2\theta$$

Radian: an angle which is subtended at the centre of a circle by an arc equal in length to the radius of the circle

Sum and difference formulas: expressions that evaluate the sine or cosine of a sum or difference

$$\sin\,(a\pm b)=\sin\,a\,\cos\,b\pm\cos\,a\,\sin\,b$$
$$\cos\,(a\pm b)=\cos\,a\,\cos\,b\pm\sin\,a\,\sin\,b$$

Trigonometric equation: an equation containing trigonometric functions in which the unknown is an angle

Trigonometric identity: an equation that equates two equivalent trigonometric expressions

Unit circle: a circle with radius equal to 1, drawn with its centre at the origin of a coordinate system, with which the ratios of the trigonometric functions of common angles can be determined

Suggested Answers

Section 1: Activity 1

1. a. $\left(\sin^2\theta\right)(\csc\,\theta)(\cot\,\theta)=\sin^2\theta\left(\frac{1}{\sin\,\theta}\right)\left(\frac{\cos\,\theta}{\sin\,\theta}\right)$

$=\cos\,\theta$

b. $\dfrac{\tan\theta\ \sin\theta}{\sec\theta}=\dfrac{\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\theta}{1}\right)}{\frac{1}{\cos\theta}}$ (quotient and reciprocal identities)

$$=\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\theta}{1}\right)\left(\frac{\cos\theta}{1}\right)$$

$$=\sin^2\theta$$

2. a.

LS	RS
$1-\sin^2\theta$	$\sin^2\theta\ \cot^2\theta$
$=\cos^2\theta$ (Pythagorean identity)	$=\sin^2\theta\bullet\dfrac{\cos^2\theta}{\sin^2\theta}$ (quotient identity)
	$=\cos^2\theta$

LS = RS

b.

LS	RS
$\sec x(1+\cos x)$	$1+\sec x$
$=\dfrac{1}{\cos x}(1+\cos x)$	$=1+\dfrac{1}{\cos x}$
$=\dfrac{1}{\cos x}+\dfrac{\cos x}{\cos x}$	
$=\dfrac{1}{\cos x}+1$	

LS = RS

c. Work with the left side of the equation.

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\sin\theta}{1-\cos\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)+\sin\theta(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$=\frac{\sin\theta-\sin\theta\ \cos\theta+\sin\theta+\sin\theta\ \cos\theta}{1-\cos^2\theta}$$

$$=\frac{2\sin\theta}{\sin^2\theta}\quad\text{(Pythagorean identity)}$$

$$=\frac{2}{\sin\theta}$$

$$=2\csc\theta\quad\text{(reciprocal identity)}$$

Therefore, LS = RS.

d. Work with the left side of the equation.

$$\frac{1+\sec\theta}{\tan\theta}-\csc\theta=\frac{1+\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}-\frac{1}{\sin\theta}\qquad\text{(reciprocal and quotient identities)}$$

$$=\frac{\left(1+\frac{1}{\cos\theta}\right)\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}$$

$$=\frac{\cos\theta+1-1}{\sin\theta}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$$=\cot\theta$$

Therefore, $\text{LS}=\text{RS}$.

e. Work with the left side of the equation.

$$\sec^2 x\sec^2 y-\tan^2 x\sec^2 y+\tan^2 x\tan^2 y-\sec^2 x\tan^2 y$$

$$=\sec^2 x\sec^2 y-\sec^2 x\tan^2 y-\tan^2 x\sec^2 y+\tan^2 x\tan^2 y\qquad\text{(Rearrange.)}$$

$$=\sec^2 x\left(\sec^2 y-\tan^2 y\right)-\tan^2 x\left(\sec^2 y-\tan^2 y\right)\qquad\text{(Factor.)}$$

$$=\sec^2 x(1)-\tan^2 x(1)\qquad\text{(Pythagorean identity)}$$

$$=\sec^2 x-\tan^2 x$$

$$=1\qquad\text{(Pythagorean identity)}$$

Therefore, $\text{LS}=\text{RS}$.

3. a. Work with the left side of the equation.

$$\frac{\csc x+\sec x}{\sin x+\cos x}=\frac{\frac{1}{\sin x}+\frac{1}{\cos x}}{\sin x+\cos x}$$

$$=\frac{\frac{\cos x+\sin x}{\sin x\ \cos x}}{\sin x+\cos x}$$

$$=\frac{\cancel{\cos x+\sin x}}{\sin x\ \cos x}\bullet\frac{1}{\cancel{\sin x+\cos x}}$$ (Multiply by the reciprocal.)

$$=\frac{1}{\sin x\ \cos x}$$

$$=\csc x\ \sec x$$ (reciprocal identities)

Therefore, $\mathrm{LS}=\mathrm{RS}$.

b. A similar identity will be $\cos x\ \cot x$.

Prove $\frac{\cos x+\cot x}{\sec x+\tan x}=\cos x\ \cot x$.

Work with the left side of the equation.

$$\frac{\cos x+\cot x}{\sec x+\tan x}=\frac{\cos x+\frac{\cos x}{\sin x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}$$

$$=\frac{\frac{\cos x\ \sin x+\cos x}{\sin x}}{\frac{1+\sin x}{\cos x}}$$

$$=\frac{\cos x\ \sin x+\cos x}{\sin x}\bullet\frac{\cos x}{1+\sin x}$$

$$=\frac{\cos x\left(\cancel{1+\sin x}\right)}{\sin x}\bullet\frac{\cos x}{\cancel{1+\sin x}}$$

$$=\frac{\cos x}{\sin x}\bullet\cos x$$

$$=\cos x\ \cot x$$

Therefore, $\mathrm{LS}=\mathrm{RS}$.

Section 1: Activity 2

1. a. $\sin 2n \cos 3n - \cos 2n \sin 3n = \sin(2n - 3n)$ (difference identity for sine)

$$= \sin(-n)$$

$$= -\sin n \quad \text{(CAST rule)}$$

b. $\cos a \cos 3a - \sin a \sin 3a = \cos(a + 3a)$ (sum formula for cosine)

$$= \cos 4a$$

c. $\dfrac{\tan 6 + \tan 4}{1 - \tan 6 \tan 4} = \tan(6 + 4)$ (sum identity for tangent)

$$= \tan 10$$

d. $\dfrac{\sin(60° + y) - \cos(y - 30°)}{\sin y}$

$$= \frac{(\sin 60° \cos y + \cos 60° \sin y) - (\cos y \cos 30° + \sin y \sin 30°)}{\sin y} \quad \text{(sum and difference formulas)}$$

$$= \frac{\frac{\sqrt{3}}{2} \cos y + \frac{1}{2} \sin y - (\cos y)\left(\frac{\sqrt{3}}{2}\right) - (\sin y)\left(\frac{1}{2}\right)}{\sin y}$$

$$= \frac{0}{\sin y}$$

$$= 0$$

2. $$\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$
$$=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$
$$=\frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}}-\dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{1+\dfrac{\sin\alpha}{\cos\alpha}\bullet\dfrac{\sin\beta}{\cos\beta}}$$
$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

3. a.

LS	RS
$\cos y-\tan x\sin y$	$\dfrac{\cos(x+y)}{\cos x}$
	$=\dfrac{\cos x\cos y-\sin x\sin y}{\cos x}$
	$=\dfrac{\cancel{\cos x}\cos y}{\cancel{\cos x}}-\dfrac{\sin x\sin y}{\cos x}$
	$=\cos y-\tan x\sin y$

LS = RS

b. Work with the left side of the equation.

$$\cos x\left[\sin\left(\frac{\pi}{2}-x\right)\right]+\sin x\left[\cos\left(\frac{\pi}{2}-x\right)\right]$$
$$=\cos x\left(\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x\right)$$
$$+\sin x\left(\cos\frac{\pi}{2}\cos x+\sin\frac{\pi}{2}\sin x\right)$$
$$=\cos x(1\cos x-0\sin x)+\sin x(0\cos x+1\sin x)$$
$$=\cos^2 x+\sin^2 x$$
$$=1$$

Therefore, LS = RS.

c. Work with the left side of the equation.

$$\cos(x+y)\cos(x-y)$$
$$=(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)$$
$$=\cos^2 x\cos^2 y-\sin^2 x\sin^2 y$$
$$=(1-\sin^2 x)\cos^2 y-\sin^2 x(1-\cos^2 y)$$
$$=\cos^2 y-\sin^2 x\cos^2 y-\sin^2 x+\sin^2 x\cos^2 y$$
$$=\cos^2 y-\sin^2 x$$

Therefore, LS = RS.

d. Work with the left side of the equation.

$$\begin{aligned}
&\tan(x+y)\tan(x-y)\\
&= \frac{\tan x+\tan y}{1-\tan x\tan y}\bullet\frac{\tan x-\tan y}{1+\tan x\tan y}\\
&= \frac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}\\
&= \frac{\frac{\sin^2 x}{\cos^2 x}-\frac{\sin^2 y}{\cos^2 y}}{1-\frac{\sin^2 x}{\cos^2 x}\bullet\frac{\sin^2 y}{\cos^2 y}}\\
&= \frac{\frac{\sin^2 x\cos^2 y-\sin^2 y\cos^2 x}{\cos^2 x\cos^2 y}}{\frac{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}{\cos^2 x\cos^2 y}}\\
&= \frac{\sin^2 x\cos^2 y-\sin^2 y\cos^2 x}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}\\
&= \frac{\sin^2 x\left(1-\sin^2 y\right)-\sin^2 y\left(1-\sin^2 x\right)}{\cos^2 x\left(1-\sin^2 y\right)-\sin^2 y\left(1-\cos^2 x\right)}\\
&= \frac{\sin^2 x-\sin^2 y}{\cos^2 x-\sin^2 y}
\end{aligned}$$

Therefore, $LS = RS$.

4. $$\begin{aligned}
\sec(a+b) &= \frac{1}{\cos(a+b)}\\
&= \frac{1}{\cos a\cos b-\sin a\sin b}\\
&= \frac{1}{\frac{1}{\sec a\sec b}-\frac{1}{\csc a\csc b}}\\
&= \frac{1}{\frac{\csc a\csc b-\sec a\sec b}{\sec a\sec b\csc a\csc b}} && \text{(A common denominator is found for the bottom.)}\\
&= \frac{\sec a\sec b\csc a\csc b}{\csc a\csc b-\sec a\sec b}
\end{aligned}$$

Section 1: Activity 3

1. a. Work with the left side of the equation.

$$\begin{aligned}
&(\sin x+\cos x)^2\\
&\quad= \sin^2 x+2\sin x\cos x+\cos^2 x\\
&\quad= \sin^2 x+\cos^2 x+2\sin x\cos x\\
&\quad= 1+2\sin x\cos x && \text{(Pythagorean identity)}\\
&\quad= 1+\sin 2x && \text{(double-angle identity)}
\end{aligned}$$

Therefore, $LS = RS$.

b. Work with the left side of the equation.

$$\begin{aligned}\sec 2x(\cos x - \sin x) &= \frac{\cos x - \sin x}{\cos 2x} \\ &= \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x} \\ &= \frac{\cancel{\cos x - \sin x}}{(\cos x + \sin x)\cancel{(\cos x - \sin x)}} \\ &= \frac{1}{\cos x + \sin x}\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

c. Work with the left side of the equation.

$$\begin{aligned}\frac{2}{1-\cos 2x} &= \frac{2}{1-\left(2\cos^2 x - 1\right)} && \text{(double-angle identity)} \\ &= \frac{2}{2-2\cos^2 x} \\ &= \frac{\cancel{2}}{\cancel{2}\left(1-\cos^2 x\right)} \\ &= \frac{1}{1-\cos^2 x} \\ &= \frac{1}{\sin^2 x} && \text{(Pythagorean identity)} \\ &= \csc^2 x\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

2. $$\begin{aligned}\tan 2a &= \tan(a+a) \\ &= \frac{\tan a + \tan a}{1-\tan a \tan a} \\ &= \frac{2\tan a}{1-\tan^2 a}\end{aligned}$$

3. Work with the left side of the equation.

$$\begin{aligned}\sin^2 \frac{x}{2}\cos^2 \frac{x}{2} &= \left(\sin \frac{x}{2}\right)^2\left(\cos \frac{x}{2}\right)^2 \\ &= \left(\pm\sqrt{\frac{1}{2}-\frac{\cos x}{2}}\right)^2\left(\pm\sqrt{\frac{1}{2}+\frac{\cos x}{2}}\right)^2 \\ &= \left(\frac{1}{2}-\frac{\cos x}{2}\right)\left(\frac{1}{2}+\frac{\cos x}{2}\right) \\ &= \frac{1}{4}-\frac{\cos^2 x}{4} \\ &= \frac{1-\cos^2 x}{4} \\ &= \frac{\sin^2 x}{4}\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$.

Section 1: Follow-up Activities

Extra Help

1. **a.** false **b.** true **c.** false **d.** true

2. **a.** $\sin^2\theta$ **b.** $\dfrac{\sin^2 A}{\cos^2 A}$ **c.** $\sin^2\beta$

   d. $\cot^2\alpha$ **e.** $\cot^2 x$

3. **a.** $\dfrac{\cos x(\csc x+1)-\sin x(\csc x-1)}{(\csc x-1)(\csc x+1)}$

   b. $\dfrac{\cot A\cos A+(\cot A+\cos A)}{\cos A(\cot A+\cos A)}$

   c. $$\begin{aligned}&\frac{2}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}-\frac{3}{\sin\theta(\sin\theta+\cos\theta)}\\&=\frac{2\sin\theta-3(\sin\theta-\cos\theta)}{\sin\theta(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}\end{aligned}$$

4. **a.** $\dfrac{\cancel{\sin x+1}}{(\cancel{\sin x+1})(\sin x-1)}=\dfrac{1}{\sin x-1}$

   b. $$\begin{aligned}\cos A\left(\cos^2 A+\sin^2 A\right)&=\cos A(1)\\&=\cos A\end{aligned}$$

   c. $$\begin{aligned}\frac{\cos^2\beta}{\sin^2\beta}\bullet\frac{\sin\beta}{\cos\beta}&=\frac{\cos\beta}{\sin\beta}\\&=\cot\beta\end{aligned}$$

   d. $$\begin{aligned}&\cos x\csc x-\cos x+\csc x-1\\&=\cos x(\csc x-1)+(\csc x-1)\\&=(\csc x-1)(\cos x+1)\end{aligned}$$

Enrichment

1. **a.** Work with the left side of the equation.

$$\begin{aligned}&\sin(A+B)-\sin(A-B)\\&=(\sin A\cos B+\cos A\sin B)\\&\quad-(\sin A\cos B-\cos A\sin B)\\&=2\cos A\sin B\end{aligned}$$

Therefore, LS = RS.

   b. Work with the left side of the equation.

$$\begin{aligned}\frac{\cos(x-y)}{\cos x\sin y}&=\frac{\cos x\cos y+\sin x\sin y}{\cos x\sin y}\\&=\frac{\cancel{\cos x}\cos y}{\cancel{\cos x}\sin y}+\frac{\sin x\cancel{\sin y}}{\cos x\cancel{\sin y}}\\&=\cot y+\tan x\end{aligned}$$

Therefore, LS = RS.

2. **a.** $\sin(a+b+c)=\sin[(a+b)+c]$

$$=\sin(a+b)\cos c+\cos(a+b)\sin c$$

$$=(\sin a\cos b+\cos a\sin b)\cos c+(\cos a\cos b-\sin a\sin b)\sin c$$

$$=\sin a\cos b\cos c+\cos a\sin b\cos c+\cos a\cos b\sin c-\sin a\sin b\sin c$$

b. $\cos(a+b+c)=\cos[(a+b)+c]$

$$=\cos(a+b)\cos c-\sin(a+b)\sin c$$

$$=(\cos a\cos b-\sin a\sin b)\cos c-(\sin a\cos b+\cos a\sin b)\sin c$$

$$=\cos a\cos b\cos c-\sin a\sin b\cos c-\sin a\cos b\sin c-\cos a\sin b\sin c$$

c. $\tan(a+b+c)=\dfrac{\sin(a+b+c)}{\cos(a+b+c)}$

$$=\frac{\sin a\cos b\cos c+\cos a\sin b\cos c+\cos a\cos b\sin c-\sin a\sin b\sin c}{\cos a\cos b\cos c-\sin a\sin b\cos c-\sin a\cos b\sin c-\cos a\sin b\sin c}$$

3. $\tan\left(\dfrac{\pi}{2}+\theta\right)=\dfrac{\tan\frac{\pi}{2}+\tan\theta}{1-\tan\frac{\pi}{2}\tan\theta}$ $\qquad$ $\tan\left(\dfrac{\pi}{2}-\theta\right)=\dfrac{\tan\frac{\pi}{2}-\tan\theta}{1+\tan\frac{\pi}{2}\tan\theta}$

No, because $\frac{\pi}{2}=\frac{1}{0}$, and is undefined; thus, the expressions cannot be simplified, using the identities for $\tan(x+y)$ and $\tan(x-y)$.

4. Work with the left side of the equation.

$$\begin{aligned}\sin 3\theta &= \sin\ (2\theta+\theta)\\ &= \sin 2\theta\ \cos\theta + \cos 2\theta\ \sin\theta\\ &= 2\sin\theta\ \cos\theta \bullet \cos\theta + \left(1-2\sin^2\theta\right)\ \sin\theta\\ &= 2\sin\theta\ \cos^2\theta + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta\left(1-\sin^2\theta\right) + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Therefore, $LS = RS$.

5. $$\begin{aligned}\cos 3\theta &= \cos\ (2\theta+\theta)\\ &= \cos 2\theta\ \cos\theta + \sin 2\theta\ \sin\theta\\ &= \left(2\cos^2\theta - 1\right)\ \cos\theta + 2\sin\theta\ \cos\theta \bullet \sin\theta\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\ \cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta\left(1-\cos^2\theta\right)\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$\begin{aligned}\tan 3\theta &= \tan\ (2\theta+\theta)\\ &= \frac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\ \tan\theta}\\ &= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\bullet\tan\theta}\\ &= \frac{\frac{2\tan\theta + \tan\theta\left(1-\tan^2\theta\right)}{1-\tan^2\theta}}{\frac{\left(1-\tan^2\theta\right)-2\tan^2\theta}{1-\tan^2\theta}}\\ &= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\end{aligned}$$

Section 2: Activity 1

1. a. $\sqrt{2}\sin\theta = 1$, where $-2\pi \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\sin\theta = \frac{\sqrt{2}}{2}$$

Therefore, the solutions are $-\frac{7\pi}{4}$, $-\frac{5\pi}{4}$, $\frac{\pi}{4}$, and $\frac{3\pi}{4}$.

b. $\tan^2 x = 3$, where $0 \le x < 2\pi$

$\tan x = \pm\sqrt{3}$

$\boxed{\tan = \dfrac{\sin}{\cos}}$

Therefore, the solutions are $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

c. $\sin x(\cos x - 1) = 0$, where $0 \le x \le 2\pi$

$\sin x = 0$ or $\cos x - 1 = 0$

$x = 0, \pi$, and 2π $\qquad \cos x = 1$

$x = 0$ and 2π

Therefore, the solutions are 0, π, and 2π.

d. $2\cos^2\theta + 5\cos\theta - 3 = 0$, where $-2\pi \le \theta \le 2\pi$

$(2\cos\theta - 1)(\cos\theta + 3) = 0$

$2\cos\theta - 1 = 0$ or $\cos\theta + 3 = 0$

$\cos\theta = \dfrac{1}{2}$ $\qquad \cos\theta = -3$

$\theta = -\dfrac{5\pi}{3}, -\dfrac{\pi}{3}, \dfrac{\pi}{3}$, and $\dfrac{5\pi}{3}$

Since there are no values of $\cos\theta$ less than -1, then $\cos\theta = -3$ is invalid. Therefore, the solutions are $-\frac{5\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$.

e. $\sin^2 x = \sin x + 2$, where $-\pi \le x \le \pi$

$\sin^2 x - \sin x - 2 = 0$

$(\sin x + 1)(\sin x - 2) = 0$

$\sin x + 1 = 0$ or $\sin x - 2 = 0$

$\sin x = -1$ $\qquad \sin x = 2$

$x = -\dfrac{\pi}{2}$

Since there are no values of $\sin x$ that are greater than 1, then there are no valid solutions for $\sin x = 2$. Therefore, the solution is $-\frac{\pi}{2}$.

2. a. $\cos^2 x + 2\sin x - 2 = 0$, where $0 \le x \le 2\pi$

Use the Pythagorean identity to transform the equation to a single function.

$$\left(1 - \sin^2 x\right) + 2\sin x - 2 = 0$$
$$\sin^2 x - 2\sin x + 1 = 0$$
$$(\sin x - 1)(\sin x - 1) = 0$$
$$\sin x - 1 = 0$$
$$\sin x = 1$$
$$x = \frac{\pi}{2}$$

The solution is $\frac{\pi}{2}$.

b.

$$\frac{\cos x+2}{\sec x}=3, \text{ where } 0\le x\le 2\pi$$
$$\cos x+2=3\sec x$$
$$\cos x+2=\frac{3}{\cos x}$$
$$\cos^2 x+2\cos x=3$$
$$\cos^2 x+2\cos x-3=0$$
$$(\cos x-1)(\cos x+3)=0$$
$$\cos x-1=0 \quad \text{or} \quad \cos x+3=0$$
$$\cos x=1 \qquad \cos x=-3$$
$$x=0 \text{ and } 2\pi$$

Since there are no values of $\cos x$ less than -1, then there are no possible solutions for $\cos x=-3$. Therefore, the solutions are $x=0$ and 2π.

c. Sine and cosine are the same when $x=\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

You may also look at the equation like the following:

$$\sin x=\cos x$$
$$\frac{\sin x}{\cos x}=1$$
$$\tan x=1$$
$$x=\frac{\pi}{4} \text{ and } \frac{5\pi}{4}$$

Therefore, the solutions are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

d.

$$\sec x=2\cos x+2, \text{ where } 0\le x\le 2\pi$$
$$\frac{1}{\cos x}=2\cos x+2$$
$$1=2\cos^2 x+2\cos x$$
$$2\cos^2 x+2\cos x-1=0$$

Use the quadratic formula.

$$\cos x=\frac{-2\pm\sqrt{4+4(2)(1)}}{2(2)}$$
$$\cos x=\frac{-2\pm\sqrt{12}}{4}$$
$$\cos x=\frac{-2\pm2\sqrt{3}}{4}$$

$$\cos x=\frac{-1+\sqrt{3}}{2} \quad \text{or} \quad \cos x=\frac{-1-\sqrt{3}}{2}$$
$$\cos x=0.366\,025\,403 \qquad x=\cos^{-1}\left(\frac{-1-\sqrt{3}}{2}\right)$$
$$x\doteq 1.2 \text{ rad}$$

Since no solution exists for $x=\cos^{-1}\left(\frac{-1-\sqrt{3}}{2}\right)$, the solution is approximately 1.2 rad.

A second solution is $2\pi-1.2$ or approximately 5.1 rad.

3. **a.** $\sin x = -\frac{1}{2}$ at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

The angles for $3x$ would be $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{7\pi}{6}+2\pi$, $\frac{11\pi}{6}+2\pi$, $\frac{7\pi}{6}+4\pi$, and $\frac{11\pi}{6}+4\pi$.

$$\therefore\ 3x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \text{ and } \frac{35\pi}{6}$$
$$x = \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \text{ and } \frac{35\pi}{18}$$

Therefore, the angles that satisfy the condition are $\frac{7\pi}{18}$, $\frac{11\pi}{18}$, $\frac{19\pi}{18}$, $\frac{23\pi}{18}$, $\frac{31\pi}{18}$, and $\frac{35\pi}{18}$.

b. $\tan 2x + 1 = 0$

$\tan 2x = -1$

$\tan x = -1$ at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

The angles for $2x$ would be $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, $\frac{3\pi}{4}+2\pi$, and $\frac{7\pi}{4}+2\pi$.

$$\therefore\ 2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \text{ and } \frac{15\pi}{4}$$
$$x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \text{ and } \frac{15\pi}{8}$$

Therefore, $x = \frac{3\pi}{8}$, $\frac{7\pi}{8}$, $\frac{11\pi}{8}$, and $\frac{15\pi}{8}$.

c. $\cos 4x - 1 = 0$

$\cos 4x = 1$

$\cos x = 1$ at 0 and 2π.

The angles for $4x$ would be 0, 2π, $0+2\pi$, $2\pi+2\pi$, $0+4\pi$, $2\pi+4\pi$, $0+6\pi$, and $2\pi+6\pi$.

Eliminate the repetitions.

$$\therefore\ 4x = 0,\ 2\pi,\ 4\pi,\ 6\pi, \text{ and } 8\pi$$
$$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \text{ and } 2\pi$$

Therefore, $x = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π.

d. $2\sin x \cos x = 0$

$\sin 2x = 0$ (double-angle identity)

$\sin x = 0$ at 0, π, and 2π.

The angles for $2x$ would be 0, π, 2π, $0+2\pi$, $\pi+2\pi$, and $2\pi+2\pi$.

$$\therefore\ 2x = 0,\ \pi,\ 2\pi,\ 3\pi, \text{ and } 4\pi$$
$$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \text{ and } 2\pi$$

Therefore, the solutions are 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π.

Section 2: Activity 2

1. $$\begin{aligned} 2\cos^2 x - 1 &= 0 \\ 2\cos^2 x &= 1 \\ \cos^2 x &= \frac{1}{2} \\ \cos x &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

For the interval $[0, 2\pi]$, cosine is $\pm\frac{1}{\sqrt{2}}$ at $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

Thus, the general solutions are $\frac{\pi}{4}+2n\pi$, $\frac{3\pi}{4}+2n\pi$, $\frac{5\pi}{4}+2n\pi$, and $\frac{7\pi}{4}+2n\pi$, where n is an integer.

2. $$\begin{aligned} 2\sin^2\theta - \sin\theta &= 1 \\ 2\sin^2\theta - \sin\theta - 1 &= 0 \\ (2\sin\theta+1)(\sin\theta-1) &= 0 \end{aligned}$$

Determine the values of θ for the interval $[0, 2\pi]$.

$$\begin{aligned} 2\sin\theta+1 &= 0 & \text{or} \quad \sin\theta - 1 &= 0 \\ \sin\theta &= -\frac{1}{2} & \sin\theta &= 1 \\ \theta &= \frac{7\pi}{6} \text{ and } \frac{11\pi}{6} & \theta &= \frac{\pi}{2} \end{aligned}$$

Thus, the general solutions are $\frac{\pi}{2}+2n\pi$, $\frac{7\pi}{6}+2n\pi$, and $\frac{11\pi}{6}+2n\pi$, where n is an integer.

3. $$\begin{aligned} \sec x \csc x &= 2\csc x \\ \frac{1}{\cos x}\bullet\frac{1}{\sin x} &= \frac{2}{\sin x} \\ 1 &= 2\cos x \quad \text{(Multiply by } \cos x \sin x\text{.)} \\ \cos x &= \frac{1}{2} \end{aligned}$$

For the interval $[0, 2\pi]$, $x=\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

Thus, the general solutions are $\frac{\pi}{3}+2n\pi$ and $\frac{5\pi}{3}+2n\pi$, where n is an integer.

4. $$\begin{aligned} \cos^2 x + \sin x &= 1 \\ (1-\sin^2 x)+\sin x - 1 &= 0 \\ \sin^2 x - \sin x &= 0 \\ \sin x(\sin x - 1) &= 0 \end{aligned}$$

Determine the values of x for the interval $[0, 2\pi]$.

$$\begin{aligned} \sin x &= 0 & \text{or} \quad \sin x - 1 &= 0 \\ x &= 0,\ \pi, \text{ and } 2\pi & \sin x &= 1 \\ & & x &= \frac{\pi}{2} \end{aligned}$$

Thus, the general solutions are $0+2n\pi$, $\frac{\pi}{2}+2n\pi$, and $\pi+2n\pi$, where n is an integer. These may be combined as $n\pi$ and $\frac{\pi}{2}+2n\pi$ where n is an integer.

5.

$$\cos\frac{x}{2}-\cos x=1$$

$$\cos\frac{x}{2}-\left(2\cos^2\frac{x}{2}-1\right)-1=0$$

(The double-angle identity is used to change x to $\frac{x}{2}$, since x is double $\frac{x}{2}$.)

$$\cos\frac{x}{2}-2\cos^2\frac{x}{2}=0$$

$$\cos\frac{x}{2}\left(1-2\cos\frac{x}{2}\right)=0$$

$$\cos\frac{x}{2}=0 \quad\text{or}\quad 1-2\cos\frac{x}{2}=0$$

$$\cos\frac{x}{2}=\frac{1}{2}$$

Cosine is 0 at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; and cosine is $\frac{1}{2}$ at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$$\therefore\ \frac{x}{2}=\frac{\pi}{3},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \text{and}\ \frac{5\pi}{3}$$

Therefore, $x=\frac{2\pi}{3}$, π, 3π, and $\frac{10\pi}{3}$.

Thus, the general solutions are $(2n-1)\pi$, $\frac{2\pi}{3}+4n\pi$, and $\frac{10\pi}{3}+4n\pi$, where n is an integer.

Section 2: Follow-up Activities

Extra Help

1. a. $2\sin x-\sqrt{3}=0$, where $0\le x\le\pi$

$$2\sin x=\sqrt{3}$$

$$\sin x=\frac{\sqrt{3}}{2}$$

$$x=\frac{\pi}{3}\ \text{and}\ \frac{2\pi}{3}$$

b. $3\cot x+3=0$, where $0\le x\le 2\pi$

$$3\cot x=-3$$

$$\cot x=-1$$

$$x=\frac{3\pi}{4}\ \text{and}\ \frac{7\pi}{4}$$

2. $-5(\sin x+1)+3=0$, where $[0,2\pi]$

$$\sin x+1=\frac{3}{5}$$

$$\sin x=-\frac{2}{5}$$

Sine is negative in the third and fourth quadrants.

(+/- 2 ÷ 5) INV sin

-0.411516846

Therefore, $x_1 \doteq -0.41$ or $2\pi + (-0.41) \doteq 5.871\,668\,461$ (a fourth quadrant angle) and $x_2 = \pi - (-0.41) \doteq 3.553\,1095$ (a third quadrant angle).

Therefore, the solutions are approximately 3.55 rad and 5.87 rad.

3. **a.** Since an interval is not specified, you must find the general solutions.

$$4\cos^2\theta - 3 = 0, \text{ where } [0, 2\pi]$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ and } \frac{11\pi}{6}$$

Therefore, the general solutions are $\frac{\pi}{6} + 2n\pi$, $\frac{5\pi}{6} + 2n\pi$, $\frac{7\pi}{6} + 2n\pi$, and $\frac{11\pi}{6} + 2n\pi$, where n is an integer.

b. $\sin x \cos x - \sin x = 0$, where $\frac{\pi}{2} \le x \le 2\pi$

$$\sin x(\cos x - 1) = 0$$

$\sin x = 0$ or $\cos x - 1 = 0$

$x = 0, \pi$, and 2π $\quad$ $\cos x = 1$

$x = 2\pi$

The solutions are 0, π, and 2π.

c. $2\sin^2\theta - 1 = \cos\theta$, where $-2\pi \le \theta \le 2\pi$

$$2\left(1 - \cos^2\theta\right) - \cos\theta - 1 = 0$$

$$2 - 2\cos^2\theta - \cos\theta - 1 = 0$$

$$-2\cos^2\theta - \cos\theta + 1 = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\cos\theta = \frac{1}{2}$ $\quad$ $\cos\theta = -1$

$\theta = -\pi$ and π

$\theta = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, and $\frac{5\pi}{3}$

Therefore, the solutions are $-\frac{5\pi}{3}$, $-\pi$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

d. $\sqrt{2}\sin 2\theta = 1$, where $0 \le \theta \le 2\pi$

$$\sin 2\theta = \frac{1}{\sqrt{2}}$$

$$\sin 2\theta = \frac{\sqrt{2}}{2}$$

Sine is $\frac{\sqrt{2}}{2}$ when θ is $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{\pi}{4}+2\pi$, and $\frac{3\pi}{4}+2\pi$.

$$\therefore\ 2\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ and } \frac{11\pi}{4}$$

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \text{ and } \frac{11\pi}{8}$$

e. $4\sin^2 3x - 1 = 0$, where $0 \le \theta \le 2\pi$

$$\sin^2 3x = \frac{1}{4}$$

$$\sin 3x = \pm\frac{1}{2}$$

Sine is $\pm\frac{1}{2}$ when $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

$$3x = \frac{\pi}{6}, \frac{\pi}{6}+2\pi, \frac{\pi}{6}+4\pi$$

$$3x = \frac{5\pi}{6}, \frac{5\pi}{6}+2\pi, \frac{5\pi}{6}+4\pi$$

$$3x = \frac{7\pi}{6}, \frac{7\pi}{6}+2\pi, \frac{7\pi}{6}+4\pi$$

$$3x = \frac{11\pi}{6}, \frac{11\pi}{6}+2\pi, \frac{11\pi}{6}+4\pi$$

$$\therefore\ 3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}, \text{ and } \frac{35\pi}{6}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}, \text{ and } \frac{35\pi}{18}$$

Enrichment

1. a. Convert 3:00 P.M. to the 24-hour clock.

$$3{:}00 \text{ P.M.} = (3+12) = 15.0 \text{ h}$$

$$h = 2\cos\left(2\pi\frac{(15.0-5.2)}{12.4}\right)+6$$

$$\doteq 6.501\,305\,064$$

At 3:00 P.M. the water is approximately 6.5 m deep.

b. $$2\cos\left(2\pi\frac{(t-5.2)}{12.4}\right)+6=7$$

$$2\cos\left(2\pi\frac{(t-5.2)}{12.4}\right)=1$$

$$\cos\left(2\pi\frac{(t-5.2)}{12.4}\right)=\frac{1}{2}$$

$$2\pi\frac{(t-5.2)}{12.4}\doteq 1.047\,197\,551$$

$$2\pi(t-5.2)\doteq 12.985\,249\,63$$

$$t-5.2\doteq 2.066\,666\,667$$

$$t\doteq 7.266\,666\,667\text{ h}$$

Convert the decimal value to minutes.

$$0.266\,666\,667\times 60\doteq 16\text{ min}$$

The water is 7 m deep at about 7:16 A.M.

2. Convert 8:30 P.M. to the 24-hour clock.

$$8:30\text{ P.M.} = 8.5+12$$
$$= 20.5\text{ h}$$

$$-2.5\cos\frac{2\pi}{365}(d+10)+18.5=20.5$$

$$-2.5\cos\frac{2\pi}{365}(d+10)=2.0$$

$$\cos\frac{2\pi}{365}(d+10)=-\frac{2.0}{2.5}$$

$$\frac{2\pi}{365}(d+10)\doteq 2.498\,091\,545$$

$$d\doteq 135.118\,021\text{ days}$$

This occurs 27 days before June 21—the longest day. Therefore, on May 26 the sun sets at 8:30 P.M. The sun would also set at 8:30 P.M. 27 days after June 21. Therefore, on July 18 the sun also sets at 8:30 P.M.

Section 3: Activity 1

1. a. $$\lim_{x\to 0}\frac{\sin 5x}{x}=\lim_{5x\to 0}\frac{5\sin 5x}{5x}$$
$$=5\bullet\lim_{5x\to 0}\frac{\sin 5x}{5x}$$
$$=5(1)$$
$$=5$$

b. $$\lim_{x\to 0}\frac{\sin \frac{1}{2}x}{x} = \lim_{\frac{1}{2}x\to 0}\frac{\frac{1}{2}\sin\frac{1}{2}x}{\frac{1}{2}x}$$
$$= \frac{1}{2}\bullet \lim_{\frac{1}{2}x\to 0}\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}$$
$$= \frac{1}{2}(1)$$
$$= \frac{1}{2}$$

c. $$\lim_{x\to 0}\frac{\sin ax}{x} = \lim_{ax\to 0}\frac{a\sin ax}{ax}$$
$$= a\bullet \lim_{ax\to 0}\frac{\sin ax}{ax}$$
$$= a(1)$$
$$= a$$

d. $$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} = \frac{\lim\limits_{3x\to 0} 3x\left(\frac{\sin 3x}{3x}\right)}{\lim\limits_{5x\to 0} 5x\left(\frac{\sin 5x}{5x}\right)}$$
$$= \frac{3}{5}\bullet\frac{\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}}{\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}$$
$$= \frac{3}{5}\left(\frac{1}{1}\right)$$
$$= \frac{3}{5}$$

e. $$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \frac{\lim\limits_{ax\to 0} ax\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{bx\to 0} bx\left(\frac{\sin bx}{bx}\right)}$$
$$= \frac{a}{b}\left(\frac{\lim\limits_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{bx\to 0}\left(\frac{\sin bx}{bx}\right)}\right)$$
$$= \frac{a}{b}\left(\frac{1}{1}\right)$$
$$= \frac{a}{b}$$

f. $$\lim_{x\to 0} x\cos x = \lim_{x\to 0} x\bullet \lim_{x\to 0}\cos x$$
$$= 0(1)$$
$$= 0$$

g. $$\lim_{x\to 0} x\csc x = \lim_{x\to 0}\frac{x}{\sin x}$$
$$= 1$$

h. $$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta}$$
$$=\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta}\bullet\frac{1}{\theta}$$
$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\bullet\frac{1}{\cos\theta}$$
$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\bullet\lim_{\theta\to 0}\sec\theta$$
$$=1(1)$$
$$=1$$

i. $$\lim_{x\to 0}\frac{\tan 2x}{3x}=\lim_{x\to 0}\frac{1}{3}\bullet\frac{\tan 2x}{x}$$
$$=\frac{1}{3}\bullet\lim_{2x\to 0}\frac{2\tan 2x}{2x}$$
$$=\frac{2}{3}\bullet\lim_{2x\to 0}\frac{\tan 2x}{2x}$$
$$=\frac{2}{3}(1)\quad\left(\lim_{x\to 0}\frac{\tan x}{x}=1\text{ as proven in question 1.h.}\right)$$
$$=\frac{2}{3}$$

2. $$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}=\lim_{x\to 0}\frac{\sin^2 x}{x^2}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\bullet\frac{\sin x}{x}$$
$$=1$$

3. $$\lim_{\theta\to 0}\frac{\cos\theta}{\theta}$$

Find the limit numerically using a table of values.

Approaching 0 from the left

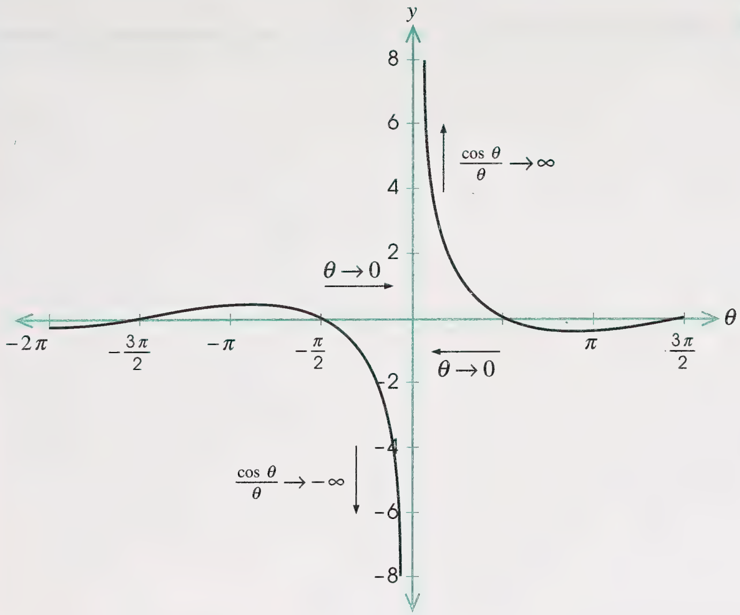
θ	$\frac{\cos\theta}{\theta}$
−0.5	−1.755 165
−0.3	−3.184 455
−0.1	−9.950 042
−0.05	−19.975 005
−0.03	−33.318 334
−0.01	−99.995 000
−0.001	−999.9995

Approaching 0 from the right

θ	$\frac{\cos\theta}{\theta}$
0.5	1.755 165
0.3	3.184 455
0.1	9.950 042
0.05	19.975 005
0.03	33.318 334
0.01	99.995 000
0.001	999.9995

The table of values indicates that the magnitude of $\frac{\cos\theta}{\theta}$ increases as θ approaches 0 from the left or right. This means that the function does not have a limit; thus, it is undefined.

Look at a graph of the function to confirm this.



The graph of the function $y = \frac{\cos\theta}{\theta}$ confirms that its limit is undefined.

Section 3: Activity 2

1. a.

$$\begin{aligned}
\lim_{x\to 0} x \cot 3x &= \lim_{x\to 0} x \bullet \frac{\cos 3x}{\sin 3x} \\
&= \lim_{x\to 0} \frac{x}{\sin 3x} \bullet \lim_{x\to 0} \cos 3x \\
&= \lim_{x\to 0} \frac{1}{3} \bullet \frac{3x}{\sin 3x} \bullet \cos 3(0) \\
&= \frac{1}{3} \bullet \lim_{x\to 0} \frac{3x}{\sin 3x} \bullet \cos 0 \\
&= \frac{1}{3} \bullet 1 \bullet 1 \\
&= \frac{1}{3}
\end{aligned}$$

b.

$$\begin{aligned}
\lim_{x\to 0} \frac{x - \tan x}{\sin x} &= \lim_{x\to 0} \left(\frac{x - \frac{\sin x}{\cos x}}{\sin x} \right) \\
&= \lim_{x\to 0} \left(\frac{\frac{x\cos x - \sin x}{\cos x}}{\sin x} \right) \\
&= \lim_{x\to 0} \left(\frac{x\cos x - \sin x}{\sin x \cos x} \right) \\
&= \lim_{x\to 0} \left(\frac{x\cos x}{\cos x \sin x} - \frac{\sin x}{\cos x \sin x} \right) \\
&= \lim_{x\to 0} \frac{x}{\sin x} - \lim_{x\to 0} \frac{1}{\cos x} \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

c. $$\lim_{x\to\frac{\pi}{3}} \frac{\sin x}{2x} = \lim_{x\to\frac{\pi}{3}} \frac{\sin \frac{\pi}{3}}{2\left(\frac{\pi}{3}\right)}$$
$$= \frac{\frac{\sqrt{3}}{2}}{\frac{2\pi}{3}}$$
$$= \frac{\sqrt{3}}{2} \bullet \frac{3}{2\pi}$$
$$= \frac{3\sqrt{3}}{4\pi}$$

d. $$\lim_{x\to-\frac{\pi}{2}} x^2 \cos^3 x = \lim_{x\to-\frac{\pi}{2}} (\cos x)^3 x^2$$
$$= \left(\cos\left(-\frac{\pi}{2}\right)\right)^3 \left(-\frac{\pi}{2}\right)^2$$
$$= (0)^3 \left(-\frac{\pi}{2}\right)^2$$
$$= 0$$

e. $$\lim_{x\to\frac{5\pi}{3}} \sqrt{\cos x} = \sqrt{\cos \frac{5\pi}{3}}$$
$$= \sqrt{\frac{1}{2}} \text{ or } \frac{\sqrt{2}}{2}$$

f. $$\lim_{x\to 2\pi} \frac{1-\sin x}{\cos x} = \lim_{x\to 2\pi} \frac{1-\sin x}{\cos x} \bullet \frac{1+\sin x}{1+\sin x}$$
$$= \lim_{x\to 2\pi} \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$
$$= \lim_{x\to 2\pi} \frac{\cos^2 x}{\cos x(1+\sin x)}$$
$$= \lim_{x\to 2\pi} \frac{\cos x}{1+\sin x}$$
$$= \frac{1}{1+0}$$
$$= 1$$

g. $$\lim_{x\to 0} \frac{\cos x-1}{3x} = \lim_{x\to 0} \frac{\cos x-1}{3x} \bullet \frac{\cos x+1}{\cos x+1}$$
$$= \lim_{x\to 0} \frac{\cos^2 x-1}{3x(\cos x+1)}$$
$$= \lim_{x\to 0} \frac{-\sin^2 x}{3x(\cos x+1)}$$
$$= \frac{1}{3} \bullet \lim_{x\to 0} \frac{\sin x}{x} \bullet \lim_{x\to 0} \frac{-\sin x}{\cos x+1}$$
$$= \frac{1}{3}(1)\left(\frac{-\sin 0}{\cos 0+1}\right)$$
$$= \frac{1}{3}\left(\frac{0}{1+1}\right)$$
$$= 0$$

h. $$\lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{2\sin x \cos x}{\frac{\sin x}{\cos x}}$$

$$= 2 \bullet \lim_{x\to\frac{\pi}{2}} \frac{\sin x \cos^2 x}{\sin x}$$

$$= 2 \bullet \lim_{x\to\frac{\pi}{2}} \cos^2 x$$

$$= 2\left(\cos\frac{\pi}{2}\right)^2$$

$$= 2(1)^2$$

$$= 2$$

i. $$\lim_{x\to\pi} \frac{\tan x - \sec x}{\sec x} = \lim_{x\to\pi} \frac{\frac{\sin x}{\cos x} - \frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \lim_{x\to\pi} \frac{\sin x - 1}{\cos x} \bullet \frac{\cos x}{1}$$

$$= \lim_{x\to\pi} (\sin x - 1)$$

$$= \sin \pi - 1$$

$$= 0 - 1$$

$$= -1$$

2. a. $$\lim_{x\to\pi} \frac{\sin x}{x-\pi}$$

$$x \to \pi$$

$$x - \pi \to 0$$

$\sin x = \sin(\pi - x)$ (CAST rule)

$$= -\sin(x-\pi)$$

$$\therefore \lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{x\to\pi} \frac{-\sin(x-\pi)}{x-\pi}$$

$$= -1$$

b. $$\lim_{x\to\frac{\pi}{2}} \frac{\cot x}{\frac{\pi}{2} - x}$$

Let $t = \frac{\pi}{2} - x$

When $x \to \frac{\pi}{2}$,

$$\left(\frac{\pi}{2} - x\right) \to 0$$

or $t \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\frac{\pi}{2} - x} = \lim_{t \to 0} \frac{\cot\left(\frac{\pi}{2} - t\right)}{t} \text{ since } x = \frac{\pi}{2} - t$$

$$= \lim_{t \to 0} \frac{\cos\left(\frac{\pi}{2} - t\right)}{\sin\left(\frac{\pi}{2} - t\right)} \bullet \frac{1}{t}$$

$$= \lim_{t \to 0} \frac{\sin t}{\cos t} \bullet \frac{1}{t}$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \bullet \lim_{t \to 0} \frac{1}{\cos t}$$

$$= 1(1)$$

$$= 1$$

Section 3: Follow-up Activities

Extra Help

1. a. $$\lim_{x \to 0} \frac{\sin x}{1 + \cos x} = \frac{\sin 0}{1 + \cos 0}$$

$$= \frac{0}{1+1}$$

$$= 0$$

b. $$\lim_{x \to 0} \frac{x + \tan x}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} + \frac{\tan x}{\sin x}$$

$$= \lim_{x \to 0} \frac{x}{\sin x} + \lim_{x \to 0} \frac{\sin x}{\cos x} \bullet \frac{1}{\sin x}$$

$$= 1 + \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 + \frac{1}{\cos 0}$$

$$= 1 + 1$$

$$= 2$$

c. $$\lim_{x \to 0} \frac{\sin^2 2x}{x^2} = \lim_{x \to 0} \frac{\sin 2x}{x} \bullet \frac{\sin 2x}{x}$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{x} \bullet \frac{2 \sin x \cos x}{x}$$

$$= 4 \bullet \lim_{x \to 0} \frac{\sin x}{x} \bullet \cos x \bullet \frac{\sin x}{x} \bullet \cos x$$

$$= 4(1)(\cos 0)(1)(\cos 0)$$

$$= 4(1)(1)(1)(1)$$

$$= 4$$

d. $$\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}=\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{2\left(\frac{x}{2}\right)} \qquad \text{(Since } x\to 0,\ \tfrac{x}{2}\to 0 \text{ also.)}$$

$$=\lim_{\frac{x}{2}\to 0}\frac{1}{2}\bullet\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=\frac{1}{2}\bullet\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=\frac{1}{2}(1)$$

$$=\frac{1}{2}$$

e. $$\lim_{x\to 0}\frac{\csc 2x}{\cot x}=\lim_{x\to 0}\frac{\tan x}{\sin 2x} \qquad \text{(reciprocal identities)}$$

$$=\lim_{x\to 0}\frac{\sin x}{\cos x}\bullet\frac{1}{2\sin x\cos x}$$

$$=\frac{1}{2}\left(\frac{1}{(\cos 0)^2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{1}\right)$$

$$=\frac{1}{2}$$

2. Let $\theta=\frac{1}{x}$ or $x=\frac{1}{\theta}$

$\therefore$ As $x\to\infty$, $\theta\to 0$

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$=1$$

Enrichment

1. Work with the left side of the equation.

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{x} \qquad \text{(double-angle identity)}$$

$$=\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{x}$$

$$=\lim_{\frac{x}{2}\to 0}\frac{-2\sin\frac{x}{2}\sin\frac{x}{2}}{2\left(\frac{x}{2}\right)}$$

$$=\frac{-2}{2}\bullet\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\bullet\sin\frac{x}{2}$$

$$=(-1)(1)\sin\left(\frac{0}{2}\right)$$

$$=(-1)(0)$$

$$=0$$

Therefore, LS = RS.

2. $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{x^2}$$
$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{4\left(\frac{x^2}{4}\right)}$$
$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{4\left(\frac{x}{2}\right)^2}$$
$$= \lim_{\frac{x}{2}\to 0}\frac{1\sin\frac{x}{2}\bullet\sin\frac{x}{2}}{2\bullet\frac{x}{2}\bullet\frac{x}{2}}$$
$$= \frac{1}{2}\bullet 1\bullet 1$$
$$= \frac{1}{2}$$

Section 4: Activity 1

1. a. $y = 3\sin x$
$$\frac{dy}{dx} = 3\bullet\frac{d}{dx}(\sin x)$$
$$= 3\cos x$$

b. $y = 4\sin x + x^2$
$$\frac{dy}{dx} = 4\bullet\frac{d}{dx}(\sin x) + \frac{d}{dx}\left(x^2\right)$$ (The derivative of a sum is the sum of the derivatives.)
$$= 4\cos x + 2x$$

c. $y = \sin 4x$
$$\frac{dy}{dx} = (\cos 4x)\bullet\frac{d}{dx}(4x)$$ (chain rule)
$$= (\cos 4x)\bullet 4$$
$$= 4\cos 4x$$

d. $y = \frac{x^2}{\sin x}$
$$\frac{dy}{dx} = \frac{\sin x\bullet\frac{d}{dx}\left(x^2\right) - x^2\bullet\frac{d}{dx}(\sin x)}{\sin^2 x}$$ (quotient rule)
$$= \frac{(\sin x)2x - x^2\cos x}{\sin^2 x}$$
$$= \frac{2x\sin x - x^2\cos x}{\sin^2 x}$$

e. $y = \left(2+x^2\right)\sin x$
$$\frac{dy}{dx} = \sin x\bullet\frac{d}{dx}\left(2+x^2\right) + \left(2+x^2\right)\bullet\frac{d}{dx}(\sin x)$$
$$= (\sin x)\bullet 2x + \left(2+x^2\right)\bullet\cos x$$
$$= 2x\sin x + 2\cos x + x^2\cos x$$

f. $y = \sin^3 x$

$$y = (\sin x)^3$$

$$\frac{dy}{dx} = 3\sin^2 x \bullet \frac{d}{dx}(\sin x)$$

$$= 3\sin^2 x \cos x$$

g. $y = \sin 2x \sin 3x$

$$\frac{dy}{dx} = \sin 3x \bullet \frac{d}{dx}(\sin 2x) + \sin 2x \bullet \frac{d}{dx}(\sin 3x)$$

$$= (\sin 3x)(\cos 2x)(2) + (\sin 2x)(\cos 3x)(3)$$

$$= 2\sin 3x \cos 2x + 3\sin 2x \cos 3x$$

h. $y = \sin \sqrt{x}$

$$y = \sin x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \left(\cos \sqrt{x}\right) \bullet \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \left(\cos \sqrt{x}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

i. $y = \dfrac{\sin 2x}{\sin 3x}$

$$\frac{dy}{dx} = \frac{\sin 3x \bullet \frac{d}{dx}(\sin 2x) - \sin 2x \bullet \frac{d}{dx}(\sin 3x)}{\sin^2 3x}$$

$$= \frac{2\sin 3x \cos 2x - 3\sin 2x \cos 3x}{\sin^2 3x}$$

2. a. $y = \cos 2x$

$$\frac{dy}{dx} = (-\sin 2x) \bullet \frac{d}{dx}(2x)$$

$$= (-\sin 2x) \bullet 2$$

$$= -2\sin 2x$$

b. $y = \cos\left(2x - \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \left[-\sin\left(2x - \frac{\pi}{2}\right)\right] \bullet \frac{d}{dx}\left(2x - \frac{\pi}{2}\right) \qquad \text{(chain rule)}$$

$$= -\sin\left(2x - \frac{\pi}{2}\right) \bullet 2$$

$$= -2\sin\left(2x - \frac{\pi}{2}\right)$$

c. $y = \dfrac{\cos x}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \bullet \frac{d}{dx}(\cos x) - \cos x \bullet \frac{d}{dx}\left(x^2\right)}{x^4} \quad \text{(quotient rule)}$$

$$= \frac{x^2 \bullet (-\sin x) - \cos x \bullet 2x}{x^4}$$

$$= \frac{-x^2 \sin x - 2x \cos x}{x^4}$$

$$= \frac{x(-x \sin x - 2 \cos x)}{x^4}$$

$$= \frac{-x \sin x - 2 \cos x}{x^3}$$

d. $y = \sin x \cos x$

$$\frac{dy}{dx} = \cos x \bullet \frac{d}{dx}(\sin x) + \sin x \bullet \frac{d}{dx}(\cos x) \quad \text{(product rule)}$$

$$= \cos x \bullet \cos x + \sin x \bullet (-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

e. $y = \dfrac{\cos^2 x}{\sin^2 x}$

$$\frac{dy}{dx} = \frac{\sin^2 x \bullet \frac{d}{dx}(\cos x)^2 - \cos^2 x \bullet \frac{d}{dx}(\sin x)^2}{\sin^4 x}$$

$$= \frac{\sin^2 x \bullet 2 \cos x \bullet (-\sin x) - \cos^2 x \bullet 2 \sin x \bullet \cos x}{\sin^4 x}$$

$$= \frac{-2 \sin^3 x \cos x - 2 \cos^3 x \sin x}{\sin^4 x}$$

$$= \frac{\sin x\left(-2 \sin^2 x \cos x - 2 \cos^3 x\right)}{\sin^4 x}$$

$$= \frac{-2 \sin^2 x \cos x - 2 \cos^3 x}{\sin^3 x}$$

f. $y = \sin(\cos x)$

$$\frac{dy}{dx} = [\cos(\cos x)] \bullet \frac{d}{dx}(\cos x) \quad \text{(chain rule)}$$

$$= \cos(\cos x) \bullet (-\sin x)$$

$$= -\sin x \cos(\cos x)$$

3. a. $y = \dfrac{1}{\tan x + 1}$

$y = (\tan x + 1)^{-1}$ (reciprocal)

$\dfrac{dy}{dx} = -1(\tan x + 1)^{-2} \bullet \dfrac{d}{dx}(\tan x + 1)$ (chain rule)

$= -1(\tan x + 1)^{-2} \bullet \left[\dfrac{d}{dx}(\tan x) + \dfrac{d}{dx}(1)\right]$

$= -1(\tan x + 1)^{-2} \bullet \left(\sec^2 x + 0\right)$

$= \dfrac{-\sec^2 x}{(\tan x + 1)^2}$

b. $y = \tan^2 x$

$y = (\tan x)^2$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\tan x)^2$ (chain rule)

$= 2 \tan x \dfrac{d}{dx}(\tan x)$

$= 2 \tan x \sec^2 x$

c. $y = \tan x^2$

$\dfrac{dy}{dx} = \sec^2 x^2 \bullet \dfrac{d}{dx}\left(x^2\right)$

$= \left(\sec^2 x^2\right) \bullet (2x)$

$= 2x \sec^2 x^2$

d. $y = \cos(\tan x)$

$\dfrac{dy}{dx} = [-\sin(\tan x)] \bullet \dfrac{d}{dx}(\tan x)$ (chain rule)

$= -\sin(\tan x) \bullet \sec^2 x$

$= -\sec^2 x \sin(\tan x)$

e. $y = \tan \sqrt{x}$

$= \tan x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \left(\sec^2 x^{\frac{1}{2}}\right) \bullet \dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$= \sec^2 x^{\frac{1}{2}} \bullet \dfrac{1}{2} x^{-\frac{1}{2}}$

$= \dfrac{\sec^2 \sqrt{x}}{2\sqrt{x}}$

Section 4: Activity 2

1. a. Work with the left side of the equation.

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{\cos x \bullet \frac{d}{dx}(1) - 1 \bullet \frac{d}{dx}(\cos x)}{\cos^2 x} \quad \text{(quotient rule)}\\ &= \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x} \bullet \frac{\sin x}{\cos x}\\ &= \sec x \tan x\end{aligned}$$

Therefore, $LS = RS$.

b. Work with the left side of the equation.

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)\\ &= \frac{\sin x \bullet \frac{d}{dx}(\cos x) - \cos x \bullet \frac{d}{dx}(\sin x)}{\sin^2 x}\\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}\\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\ &= \frac{-\left(\sin^2 x + \cos^2 x\right)}{\sin^2 x}\\ &= \frac{-1}{\sin^2 x} \quad \text{(Pythagorean identity)}\\ &= -\csc^2 x\end{aligned}$$

Therefore, $LS = RS$.

2. a.

$$\begin{aligned}y &= \csc^2 2x\\ &= (\csc 2x)^2\\ \frac{dy}{dx} &= 2\csc 2x \bullet \frac{d}{dx}(\csc 2x)\\ &= (2\csc 2x)(-\csc 2x)(\cot 2x)(2)\\ &= -4\csc^2 2x \cot 2x\end{aligned}$$

b. $y = \sec x \tan x$

$$\frac{dy}{dx} = \sec x \bullet \frac{d}{dx}(\tan x) + \tan x \bullet \frac{d}{dx}(\sec x)$$

$$= \sec x \bullet \sec^2 x + \tan x \bullet \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

c. $y = \sec^2 x - \tan^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x)^2 - \frac{d}{dx}(\tan x)^2$$

$$= 2 \sec x \bullet \frac{d}{dx}(\sec x) - 2 \tan x \bullet \frac{d}{dx}(\tan x)$$

$$= 2 \sec x \bullet \sec x \tan x - 2 \tan x \bullet \sec^2 x$$

$$= 2 \sec^2 x \tan x - 2 \tan x \sec^2 x$$

$$= 0$$

d. $y = x^2 \cot x$

$$\frac{dy}{dx} = x^2 \bullet \frac{d}{dx}(\cot x) + \cot x \bullet \frac{d}{dx}\left(x^2\right)$$

$$= x^2\left(-\csc^2 x\right) + (\cot x)(2x)$$

$$= 2x \cot x - x^2 \csc^2 x$$

e. $y = \dfrac{\csc 2x}{2x}$

$$\frac{dy}{dx} = \frac{2x \bullet \frac{d}{dx}(\csc 2x) - \csc 2x \bullet \frac{d}{dx}(2x)}{(2x)^2}$$

$$= \frac{(2x)(-\csc 2x)(\cot 2x)(2) - (\csc 2x)(2)}{4x^2}$$

$$= \frac{-4x \csc 2x \cot 2x - 2 \csc 2x}{4x^2}$$

$$= \frac{2(-2x \csc 2x \cot 2x - \csc 2x)}{4x^2}$$

$$= \frac{-2x \csc 2x \cot 2x - \csc 2x}{2x^2}$$

3. a. $y = \sin^2 x + \tan^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\tan x)^2$$

$$= 2 \sin x \bullet \frac{d}{dx}(\sin x) + 2 \tan x \bullet \frac{d}{dx}(\tan x)$$

$$= 2 \sin x \bullet \cos x + 2 \tan x \bullet \sec^2 x$$

$$= \sin 2x + 2 \tan x \sec^2 x$$

b. $y = \dfrac{\sin x}{\sec x}$

$$\frac{dy}{dx} = \frac{\sec x \bullet \frac{d}{dx}(\sin x) - \sin x \bullet \frac{d}{dx}(\sec x)}{\sec^2 x}$$

$$= \frac{\sec x \bullet \cos x - \sin x \bullet \sec x \tan x}{\sec^2 x}$$

$$= \frac{1 - \sin x \sec x \tan x}{\sec^2 x} \qquad \text{(reciprocal identities)}$$

$$= \frac{1 - \tan^2 x}{\sec^2 x}$$

c. $y = \csc^2 x - \cot^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\csc x)^2 - \frac{d}{dx}(\cot x)^2$$

$$= 2 \csc x \bullet \frac{d}{dx}(\csc x) - 2 \cot x \bullet \frac{d}{dx}(\cot x)$$

$$= 2 \csc x(-\csc x \cot x) - 2 \cot x\left(-\csc^2 x\right)$$

$$= -2 \csc^2 x \cot x + 2 \csc^2 x \cot x$$

$$= 0$$

d. $y = (\csc x + \cot x)^2$

$$\frac{dy}{dx} = 2(\csc x + \cot x) \bullet \left[\frac{d}{dx}(\csc x) + \frac{d}{dx}(\cot x)\right]$$

$$= 2(\csc x + \cot x)\left(-\csc x \cot x - \csc^2 x\right)$$

$$= -2(\csc x + \cot x)\left(\csc x \cot x + \csc^2 x\right)$$

e. $y = 3 \csc x \cos^3 x$

$$\frac{dy}{dx} = 3 \bullet \csc x \bullet \frac{d}{dx}(\cos x)^3$$

$$+ \cos^3 x \bullet \frac{d}{dx}(3 \csc x)$$

$$= 3 \csc x \bullet 3 \cos^2 x(-\sin x) + \cos^3 x(-3 \csc x \cot x)$$

$$= 3\left[-3 \cos^2 x - \cos^3 x \bullet \csc x \bullet \frac{\cos x}{\sin x}\right]$$

$$= 3\left(-3 \cos^2 x - \frac{\cos^4 x}{\sin^2 x}\right)$$

$$= -9 \cos^2 x - 3 \cos^2 x \cot^2 x$$

Section 4: Activity 3

1. a. $y = \dfrac{\sin x + 2\cos x}{\sin x - 2\cos x}$

$$\frac{dy}{dx} = \frac{(\sin x - 2\cos x)\bullet \frac{d}{dx}(\sin x + 2\cos x) - (\sin x + 2\cos x)\bullet \frac{d}{dx}(\sin x - 2\cos x)}{(\sin x - 2\cos x)^2}$$

$$= \frac{(\sin x - 2\cos x)(\cos x - 2\sin x) - (\sin x + 2\cos x)(\cos x + 2\sin x)}{(\sin x - 2\cos x)^2}$$

$$= \frac{\left(5\sin x\cos x - 2\sin^2 x - 2\cos^2 x\right) - \left(5\sin x\cos x + 2\sin^2 x + 2\cos^2 x\right)}{(\sin x - 2\cos x)^2}$$

$$= \frac{-4\sin^2 x - 4\cos^2 x}{(\sin x - 2\cos x)^2}$$

$$= \frac{-4\left(\sin^2 x + \cos^2 x\right)}{(\sin x - 2\cos x)^2}$$

$$= \frac{-4}{(\sin x - 2\cos x)^2} \qquad \text{(Pythagorean identity)}$$

b. $y = 9x + 3\cot 3x - \cot^3 3x$

$$\frac{dy}{dx} = \frac{d}{dx}(9x) + \frac{d}{dx}(3\cot 3x) - \frac{d}{dx}(\cot 3x)^3$$

$$= 9 + 3\left(-\csc^2 3x\right)\frac{d}{dx}(3x) - \left(3\cot^2 3x\right)\frac{d}{dx}(\cot 3x)$$

$$= 9 - 9\csc^2 3x - 3\cot^2 3x\left(-\csc^2 3x\right)\frac{d}{dx}(3x)$$

$$= 9 - 9\csc^2 3x + 9\cot^2 3x \csc^2 3x$$

$$= 9\left(1 - \csc^2 3x + \cot^2 3x \csc^2 3x\right)$$

$$= 9\left[\left(1 - \csc^2 3x\right) + \cot^2 3x\left(1 + \cot^2 3x\right)\right]$$

$$= 9\left[\cancel{-\cot^2 3x} + \cancel{\cot^2 3x} + \cot^4 3x\right]$$

$$= 9\cot^4 3x$$

c. $y = \frac{u^5}{5} - \frac{u^3}{3}$, when $u = \tan x$

$$y = \frac{(\tan x)^5}{5} - \frac{(\tan x)^3}{3}$$

$$\frac{dy}{dx} = \tan^4 x \bullet \frac{d}{dx}(\tan x) - \tan^2 x \bullet \frac{d}{dx}(\tan x)$$

$$= \tan^4 x \sec^2 x - \tan^2 x \sec^2 x$$

$$= \sec^2 x\left(\tan^4 x - \tan^2 x\right)$$

$$= \left(1 + \tan^2 x\right)\left(\tan^4 x - \tan^2 x\right)$$

$$= \tan^4 x - \tan^2 x + \tan^6 x - \tan^4 x$$

$$= \tan^6 x - \tan^2 x$$

d. $y = \frac{\cos 3x}{x^2 + 2} + \tan 2x$

$$\frac{dy}{dx} = \frac{\left(x^2+2\right) \bullet \frac{d}{dx}(\cos 3x) - \cos 3x \bullet \frac{d}{dx}\left(x^2+2\right)}{\left(x^2+2\right)^2}$$

$$+ \frac{d}{dx}(\tan 2x)$$

$$= \frac{\left(x^2+2\right)(-\sin 3x)(3) - (\cos 3x)(2x)}{\left(x^2+2\right)^2} + 2\sec^2 2x$$

$$= \frac{-3\left(x^2+2\right)\sin 3x - 2x\cos 3x}{\left(x^2+2\right)^2} + 2\sec^2 2x$$

e. $y = \sqrt{\sin 2x + x} + \dfrac{\csc 3x}{x^3 + 1}$

$$y = (\sin 2x + x)^{\frac{1}{2}} + \frac{\csc 3x}{x^3 + 1}$$

$$\frac{dy}{dx} = \frac{1}{2}(\sin 2x + x)^{-\frac{1}{2}} \bullet \frac{d}{dx}(\sin 2x + x) + \frac{\left(x^3 + 1\right) \bullet \frac{d}{dx}(\csc 3x) - \csc 3x \bullet \frac{d}{dx}\left(x^3 + 1\right)}{\left(x^3 + 1\right)^2}$$

$$= \frac{(\cos 2x)\frac{d}{dx}(2x) + 1}{2\sqrt{\sin 2x + x}} + \frac{\left(x^3 + 1\right)(-\csc 3x \cot 3x)\frac{d}{dx}(3x) - (\csc 3x)3x^2}{\left(x^3 + 1\right)^2}$$

$$= \frac{2\cos 2x + 1}{2\sqrt{\sin 2x + x}} + \frac{-3\left(x^3 + 1\right)\csc 3x \cot 3x - 3x^2 \csc 3x}{\left(x^3 + 1\right)^2}$$

$$= \frac{\cos 2x + 1}{\sqrt{\sin 2x + x}} + \frac{-3\left(x^3 + 1\right)\csc 3x \cot 3x - 3x^2 \csc 3x}{\left(x^3 + 1\right)^2}$$

2. a.

$$\sin xy = x$$
$$\frac{d}{dx}(\sin xy) = \frac{d}{dx}(x)$$
$$\cos xy \bullet \frac{d}{dx}(xy) = 1 \qquad \text{(chain rule)}$$
$$\cos xy \bullet \left(x \bullet \frac{dy}{dx} + y \bullet 1 \right) = 1 \qquad \text{(product rule)}$$
$$x \cos xy \bullet \frac{dy}{dx} + y \cos xy = 1$$
$$x \cos xy \bullet \frac{dy}{dx} = 1 - y \cos xy$$
$$\frac{dy}{dx} = \frac{1 - y \cos xy}{x \cos xy}$$

b.

$$\tan(x+y) = y$$
$$\frac{d}{dx}\left[\tan(x+y)\right] = \frac{d}{dx}(y)$$
$$\sec^2(x+y) \bullet \frac{d}{dx}(x+y) = \frac{dy}{dx}$$
$$\sec^2(x+y) \bullet \left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}$$
$$\sec^2(x+y) + \sec^2(x+y) \bullet \frac{dy}{dx} = \frac{dy}{dx}$$
$$\sec^2(x+y) \bullet \frac{dy}{dx} - \frac{dy}{dx} = -\sec^2(x+y)$$
$$\frac{dy}{dx}\left[\sec^2(x+y) - 1\right] = -\sec^2(x+y)$$
$$\frac{dy}{dx} = \frac{-\sec^2(x+y)}{\sec^2(x+y) - 1}$$
$$= \frac{-\sec^2(x+y)}{\tan^2(x+y)} \qquad \text{(Pythagorean identity)}$$
$$= \frac{-1}{\cancel{\cos^2(x+y)}} \bullet \frac{\cancel{\cos^2(x+y)}}{\sin^2(x+y)}$$
$$= \frac{-1}{\sin^2(x+y)}$$
$$= -\csc^2(x+y)$$

c.

$$\begin{aligned}
y^2 - xy &= \sin x \\
\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(xy) &= \frac{d}{dx}(\sin x) \\
2y \bullet \frac{dy}{dx} - \left(x \bullet \frac{dy}{dx} + y \bullet 1\right) &= \cos x \\
2y \bullet \frac{dy}{dx} - x \bullet \frac{dy}{dx} - y &= \cos x \\
(2y - x)\frac{dy}{dx} &= \cos x + y \\
\frac{dy}{dx} &= \frac{\cos x + y}{2y - x}
\end{aligned}$$

d.

$$\begin{aligned}
2x^2 y + y^2 &= \cos x \\
2 \bullet \frac{d}{dx}\left(x^2 y\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(\cos x) \\
2\left(2x \bullet y + x^2 \bullet \frac{dy}{dx}\right) + 2y \bullet \frac{dy}{dx} &= -\sin x \\
4xy + 2x^2 \bullet \frac{dy}{dx} + 2y \bullet \frac{dy}{dx} &= -\sin x \\
\left(2x^2 + 2y\right)\frac{dy}{dx} &= -\sin x - 4xy \\
\frac{dy}{dx} &= \frac{-\sin x - 4xy}{2x^2 + 2y}
\end{aligned}$$

e.

$$\begin{aligned}
x + y &= \cot(x - y) \\
\frac{d}{dx}(x) + \frac{d}{dx}(y) &= \frac{d}{dx}\left[\cot(x - y)\right] \\
1 + \frac{dy}{dx} &= -\csc^2(x - y) \bullet \frac{d}{dx}(x - y) \\
1 + \frac{dy}{dx} &= -\csc^2(x - y) \bullet \left(1 - \frac{dy}{dx}\right) \\
1 + \frac{dy}{dx} &= -\csc^2(x - y) + \csc^2(x - y)\frac{dy}{dx} \\
\frac{dy}{dx} - \csc^2(x - y)\frac{dy}{dx} &= -\csc^2(x - y) - 1 \\
\frac{dy}{dx}\left[1 - \csc^2(x - y)\right] &= -\csc^2(x - y) - 1 \\
\frac{dy}{dx} &= \frac{-\csc^2(x - y) - 1}{1 - \csc^2(x - y)} \\
&= \frac{-\left[\csc^2(x - y) + 1\right]}{-\cot^2(x - y)} \\
&= \left[\csc^2(x - y) + 1\right] \bullet \tan^2(x - y) \\
&= \sec^2(x - y) + \tan^2(x - y)
\end{aligned}$$

f.

$$x^2 - x\cos y + y^3 = 0$$

$$\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x\cos y) + \frac{d}{dx}\left(y^3\right) = 0$$

$$2x - \left[1 \bullet \cos y + x \bullet \frac{d}{dx}(\cos y)\right] + 3y^2 \bullet \frac{dy}{dx} = 0$$

$$2x - \cos y - x\sin y \bullet \frac{dy}{dx} + 3y^2 \bullet \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-x\sin y + 3y^2\right) = \cos y - 2x$$

$$\frac{dy}{dx} = \frac{\cos y - 2x}{-x\sin y + 3y^2}$$

Section 4: Activity 4

1. a. At $x = \frac{\pi}{3}$, $y = \sin\frac{\pi}{3}$ (Read this value from the unit circle.)

$$= \frac{\sqrt{3}}{2}$$

The point of tangency is $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$$
$$= \cos x$$

$$\therefore m = \cos\frac{\pi}{3}$$
$$= \frac{1}{2}$$

Find the equation of the tangent line.

$$y - y_1 = m\left(x - x_1\right)$$

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

$$y = \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2} \quad \text{or} \quad 3x - 6y - \pi + 3\sqrt{3} = 0$$

b. At $x = \frac{\pi}{4}$, $y = \left(\tan\frac{\pi}{4}\right)^3$

$$= (1)^3$$
$$= 1$$

The point of tangency is $\left(\frac{\pi}{4}, 1\right)$.

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x)^3$$
$$= 3\tan^2 x \bullet \frac{d}{dx}(\tan x)$$
$$= 3\tan^2 x \sec^2 x$$

$$\therefore m = 3\left(\tan\frac{\pi}{4}\right)^2\left(\sec\frac{\pi}{4}\right)^2$$
$$= 3\bullet(1)^2\bullet\left(\frac{2}{\sqrt{2}}\right)^2$$
$$= 6$$

Find the equation of the tangent line.

$$y - y_1 = m\left(x - x_1\right)$$
$$y - 1 = 6\left(x - \frac{\pi}{4}\right)$$
$$y = 6x - \frac{3\pi}{2} + 1 \quad \text{or} \quad 12x - 2y - 3\pi + 2 = 0$$

c. $$\frac{dy}{dx} = \frac{d}{dx}(\sec x)$$
$$= \sec x \tan x$$

At $x = \frac{\pi}{3}$, $m = \sec\frac{\pi}{3}\tan\frac{\pi}{3}$
$$= 2\bullet\sqrt{3}$$
$$= 2\sqrt{3}$$

$$\therefore y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$
$$y = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3} + 2 \quad \text{or} \quad 6\sqrt{3}x - 3y - 2\sqrt{3}\pi + 6 = 0$$

d. At $x = \frac{2\pi}{3}$, $y = \cos 2\left(\frac{2\pi}{3}\right)$
$$= \cos\frac{4\pi}{3}$$
$$= -\frac{1}{2}$$

The point of tangency is $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos 2x)$$
$$= -\sin 2x \bullet \frac{d}{dx}(2x)$$
$$= -2\sin 2x$$

$$\therefore\ m = -2\sin 2\left(\frac{2\pi}{3}\right)$$
$$= -2\sin\frac{4\pi}{3}$$
$$= -2\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \sqrt{3}$$

Find the equation of the tangent.

$$y+\frac{1}{2} = \sqrt{3}\left(x-\frac{2\pi}{3}\right)$$
$$y = \sqrt{3}x - \frac{2\sqrt{3}}{3}\pi - \frac{1}{2} \text{ or } 6\sqrt{3}x - 6y - 4\sqrt{3}\pi - 3$$

e. $y = x - \sin x$ at $x = \frac{\pi}{2}$

At $x = \frac{\pi}{2}$, $y = \frac{\pi}{2} - \sin\frac{\pi}{2}$

$$= \frac{\pi}{2} - 1$$
$$= \frac{\pi - 2}{2}$$

The point of tangency is $\left(\frac{\pi}{2}, \frac{\pi-2}{2}\right)$.

$$\frac{dy}{dx} = \frac{d}{dx}(x) - \frac{d}{dx}(\sin x)$$
$$= 1 - \cos x$$

At $x = \frac{\pi}{2}$, $m = 1 - \cos\frac{\pi}{2}$

$$= 1 - 0$$
$$= 1$$

This slope indicates a horizontal line.

Therefore, the tangent line is defined as follows:

$$y - \frac{\pi-2}{2} = 1\left(x - \frac{\pi}{2}\right)$$
$$2y - (\pi - 2) = 2\left(x - \frac{\pi}{2}\right)$$
$$2y - \pi + 2 = 2x - \pi$$
$$2x - 2y - 2 = 0$$
$$x - y - 1 = 0$$

2. $y = \sin x$ at $x = 16.3$ (Round any required decimals to two places.)

At $x = 16.3$, $y = \sin 16.3$

$$\doteq -0.558\,052\,27$$
$$\doteq -0.56$$

Make sure the calculator is in radian mode.

The point of tangency is approximately $(16.3, -0.56)$.

$$\frac{dy}{dx}=\frac{d}{dx}(\sin x)$$
$$=\cos x$$

At $x=16.3,\ m=\cos 16.3$
$$\doteq -0.829\,805\,798$$
$$\doteq -0.83$$

The tangent has a slope of approximately -0.83, and passes through $(16.3,-0.56)$.

$$\therefore\ y+0.56 \doteq -0.83(x-16.3)$$
$$y \doteq -0.83x+13.53-0.56$$
$$\text{or}\ \ 0.83x+y-12.97 \doteq 0$$

Section 4: Follow-up Activities

Extra Help

1. a. $y=2x\cos x-2\sin x$

$$\frac{dy}{dx}=\frac{d}{dx}(2x\cos x)-\frac{d}{dx}(2\sin x)$$

The first term of the function requires the product rule because both factors are variables. In the second term you must use the constant multiple rule.

$$\frac{dy}{dx}=2x\bullet\frac{d}{dx}(\cos x)+\cos x\bullet\frac{d}{dx}(2x)-2\bullet\frac{d}{dx}(\sin x)$$
$$=2x(-\sin x)+2\cos x-2\cos x$$
$$=-2x\sin x$$

b. $y=\dfrac{\sec^7 x}{7}-\dfrac{\sec^5 x}{5}$

$$y=\frac{(\sec x)^7}{7}-\frac{(\sec x)^5}{5}$$

Use the power rule and the difference rule to find the derivative of this function.

$$\frac{dy}{dx}=\frac{\cancel{7}(\sec x)^6(\sec x\ \tan x)}{\cancel{7}}-\frac{\cancel{5}(\sec x)^4(\sec x\ \tan x)}{\cancel{5}}$$
$$=(\sec x)^6(\sec x\ \tan x)-(\sec x)^4(\sec x\ \tan x)$$
$$=\sec^7 x\ \tan x-\sec^5 x\ \tan x$$
$$=\sec^5 x\ \tan x\left(\sec^2 x-1\right)$$
$$=\sec^5 x\ \tan x\left(\tan^2 x\right)\quad \text{(Pythagorean identity)}$$
$$=\sec^5 x\ \tan^3 x$$

2. a. $y=\csc 3x+\cot 3x$

$$\frac{dy}{dx}=\frac{d}{dx}(\csc 3x)+\frac{d}{dx}(\cot 3x)$$

Apply the chain rule and the sum rule for differentiation.

$$\frac{dy}{dx} = -\csc 3x \cot 3x \bullet \frac{d}{dx}(3x) + \left(-\csc^2 3x\right) \bullet \frac{d}{dx}(3x)$$
$$= -3\csc 3x \cot 3x - 3\csc^2 x$$

Simplify.

$$\frac{dy}{dx} = -3\csc 3x(\cot 3x + \csc 3x)$$
$$= -3\left(\frac{1}{\sin 3x}\right)\left(\frac{\cos 3x}{\sin 3x} + \frac{1}{\sin 3x}\right) \quad \text{(reciprocal and quotient identities)}$$
$$= \frac{-3(\cos 3x + 1)}{\sin^2 3x}$$
$$= \frac{-3(\cos 3x + 1)}{1 - \cos^2 3x} \quad \text{(Pythagorean identity)}$$
$$= \frac{-3\cancel{(\cos 3x + 1)}}{(1 - \cos 3x)\cancel{(1 + \cos 3x)}} \quad \text{(Factor the denominator and cancel.)}$$
$$= \frac{-3}{1 - \cos 3x}$$

b. $y = \sin x(\sin x + \cos x)$

$y = \sin^2 x + \sin x \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\sin x \cos x)$$

Use the power rule, the chain rule, and the product rule.

$$\frac{dy}{dx} = 2\sin x \bullet \frac{d}{dx}(\sin x)$$
$$+ \left[\sin x \bullet \frac{d}{dx}(\cos x) + \cos x \bullet \frac{d}{dx}(\sin x)\right]$$
$$= 2\sin x \cos x + \sin x \bullet (-\sin x) + \cos x \bullet \cos x$$
$$= 2\sin x \cos x + \cos^2 x - \sin^2 x$$
$$= \sin 2x + \cos 2x \quad \text{(double-angle identities)}$$

Enrichment

1. $\frac{dy}{dx} = -2 \bullet \frac{d}{dx}(\cos x)$

$= -2 \bullet -\sin x$

$= 2\sin x$

$\frac{d^2y}{dx^2} = 2 \bullet \frac{d}{dx}(\sin x)$

$= 2\cos x$

2. $\frac{dy}{dx} = \cos x$ $\quad \frac{d^2y}{dx^2} = -\sin x$ $\quad \frac{d^3y}{dx^2} = -\cos x$

$\frac{d^4y}{dx^4} = \sin x$ $\quad \frac{d^5y}{dx^5} = \cos x$ $\quad \frac{d^6y}{dx^6} = -\sin x$

$\frac{d^7y}{dx^7} = -\cos x$ $\quad \frac{d^8y}{dx^8} = \sin x$

Notice that every fourth derivative is the same. For instance

$\frac{dy}{dx} = \frac{d^5y}{dx^5} = \frac{d^9y}{dx^9} = \ldots.$

3. $\frac{dy}{dx} = -\sin x$ $\qquad \frac{d^2y}{dx^2} = -\cos x$ $\qquad \frac{d^3y}{dx^3} = \sin x$

$\frac{d^4y}{dx^4} = \cos x$ $\qquad \frac{d^5y}{dx^5} = -\sin x$ $\qquad \frac{d^6y}{dx^6} = -\cos x$

$\frac{d^7y}{dx^7} = \sin x$ $\qquad \frac{d^8y}{dx^8} = \cos x$

As in question 2, the pattern is the same. The same derivatives recur in sets of 4.

4. $\frac{dy}{dx} = 2\cos x$ $\qquad \frac{d^2y}{dx^2} = -2\sin x$

$\frac{d^3y}{dx^3} = -2\cos x$ $\qquad \frac{d^4y}{dx^4} = 2\sin x$

$$\therefore f^{(4)}\left(\frac{\pi}{2}\right) = 2\sin\frac{\pi}{2}$$
$$= 2$$

5. $\sin y = \cos x$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$
$$\cos y \bullet \frac{dy}{dx} = -\sin x$$
$$\frac{dy}{dx} = \frac{-\sin x}{\cos y}$$

$$\frac{d^2y}{dx^2} = \frac{\cos y \bullet \frac{d}{dx}(-\sin x) - (-\sin x) \bullet \frac{d}{dx}(\cos y)}{\cos^2 y}$$
$$= \frac{\cos y(-\cos x) + \sin x(-\sin y)\left(\frac{dy}{dx}\right)}{\cos^2 y}$$
$$= \frac{-\cos x \cos y - \sin x \sin y\left(\frac{-\sin x}{\cos y}\right)}{\cos^2 y}$$
$$= \frac{\frac{-\cos x \cos^2 y + \sin^2 x \sin y}{\cos y}}{\cos^2 y}$$
$$= \frac{-\cos x \cos^2 y + \sin^2 x \sin y}{\cos^3 y}$$

Replace $\frac{dy}{dx}$ with $\frac{-\sin x}{\cos y}$.

LRDC
Producer

Mathematics 31
Student Module Booklet
Module 4

1995